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Existence of stationary solutions to coagulation equations with injection

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Tiivistelmä — Referat — Abstract

Coagulation equations are evolution equations that model the time-evolution of the size-distribution of particles in systems where colliding particles stick together, or coalesce, to form one larger particle. These equations arise in many areas of science, most prominently in aerosol physics and the study of polymers. In the former case, the colliding particles are small aerosol particles that form ever larger aerosol particles, and in the latter case, the particles are polymers of various sizes.

As the system evolves, the density of particles of a specified size changes. The rate of change is specified by two competing factors. On one hand there is a positive contribution coming from smaller particles coalescing to form particles of this specific size. On the other hand, particles of this size can coalesce with other particles to form larger particles, which contributes negatively to the density of particles of this size. Furthermore, if there is no addition of new particles into the system, then the total mass of the particles should remain constant. From these considerations, it follows that the time-evolution of the coagulation equation is specified for every particle size by a difference of two terms which preserve the total mass of the system. The physical properties of the system affect the time evolution via a coagulation kernel, which determines the rate at which particles of different sizes coalesce.

A variation of coagulation equations is achieved when we add an *injection* term to the evolution equation to account for new particles injected into the system. This results in a new evolution equation, a coagulation equation with injection, where the total mass of the system is no longer preserved, as new particles are added into the system at each point in time. Coagulation equations with injection may have non-trivial solutions that are independent of time. The existence of non-trivial stationary solutions has ramifications in aerosol physics, since these might map to observations that the particle size distribution in the air stays approximately constant.

In this thesis, it will be demonstrated, following Ferreira et al. (2019), that for any good enough injection term and for suitably picked, compactly supported coagulation kernels, there exists a stationary solution to a regularized version of the coagulation equation. This theorem, which relies heavily on functional analytic tools, is a central step in the proof that certain asymptotically well-behaved kernels have stationary solutions for any prescribed compactly supported injection term.

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Contents

1 Introduction

Consider a system of particles enclosed in a container, and suppose that colliding particles are susceptible of sticking together, forming a larger particle as a result. Such processes, where smaller particles form ever larger particles upon collisions, are known as *coagulation processes*. Interest in coagulation processes initially arose from physics and chemistry, where the mathematical model of coagulation was laid out by Marian Smoluchowski to describe the formation of colloids in liquids [29].

The study of coagulation processes is also higly relevant in aerosol physics, where one of the objects of study is the evolution of the size distribution of particles suspended in some gas. Indeed, the coagulation of small particles is one contributing factor in the formation of larger aerosol particles. The coalescing particles can eventually reach sizes where they can influence, for example, clouds or human health. Understanding how coagulation processes behave is thereby central to our understanding of the atmosphere, among other things.

In recent years, mathematical equations modeling coagulation processes have received wide interest. These *coagulation equations* model how the size distribution of particles in the system evolves as a function of time. Here, the size or mass of the particle should be interpreted abstractly to denote the value of the size variable i, but physically it can mean how many monomers the particle contains or what its physical mass is. With this in mind, let us denote by $n(i,t) \in \mathbb{R}_+$ the density of particles of size $i \in \mathbb{N}_+$ in the system at time $t \in [0,\infty)$. We also assume that the space in which the coagulation process takes place and the physical properties of this system can be encoded in some kind of a coagulation rate kernel $K(\cdot,\cdot)$, which tells the rate at which particles of different sizes coalesce in the system. Coagulation equations describe the time-evolution of the particle size distribution in terms of the coagulation rate K and the densities $n(i,t)$. A natural question to ask is then: What kind of properties do coagulation equations possess?

In the coagulation equation, the rate of change of the density of particles of size i at time t, which we denote here by $\partial_t n(i,t)$, is determined by two competing terms: one with a positive contribution and one with a negative contribution. Smaller clusters can come together to form clusters of size i, whereby $\partial_t n(i,t)$ should depend at least on the quantitites $n(j,t)$, $n(i-j,t)$, and some rate function K, where $K(j,i-j)$ tells us the pace of the coalescence of particles of size j and $i-j$. On the other hand, clusters of size i coagulate with clusters of other sizes to produce larger clusters, and this reduces the

quantity $n(i,t)$. The combination of these considerations yields the following evolution equation:

$$\begin{aligned}\partial_t n(i,t) &= \frac{1}{2}\sum_{j=1}^{i-1} K(j,i-j)n(j,t)n(i-j,t) \\ &\quad - \sum_{j=1}^{\infty} K(i,j)n(i,t)n(j,t). \end{aligned} \tag{1.1}$$

Here the coagluation rate *kernel* function $K\colon \mathbb{N}\times\mathbb{N}\to\mathbb{R}_+$ is a symmetric, positive function, which describes the rate at which particles of size i and size j coalesce to form particles of size $i+j$. The first term in (1.1) accounts for the increasing effect of smaller particles coming together to form particles of size i and the second term accounts for the decreasing effect of particles of size i coalescing with other particles. The discrepancy between the 1/2-factor in the first term and the 1-factor in the second term comes from the fact that in the first term we have to avoid double counting.

The form of equation (1.1) suggests a continuous variation, in case we want to allow the particle sizes to take arbitrary positive real values. The reason for considering any real values is that in certain application we are only interested in clusters that are large enough to be "physically relevant". In other words, we are interested in those particle sizes $x = \frac{i}{i_{\text{ref}}}$, where $i, i_{\text{ref}} \in \mathbb{N}$ and $i_{\text{ref}} \gg 1$. Then, in the large regime, where $x \sim 1$, x can take non-integer values as well [8, p. 309]. As an approximation, we assume that the values of x are continuous. The continuous version of the coagulation equation is then attained by replacing the sums in (1.1) by integrals:

$$\begin{aligned}\partial_t f(x,t) &= \frac{1}{2}\int_0^x K(x-y,y)f(x-y,t)f(y,t)\,\mathrm{d}y \\ &\quad - \int_0^\infty K(x,y)f(x,t)f(y,t)\,\mathrm{d}y. \end{aligned} \tag{1.2}$$

Of course, we must also state what we mean by the function $f(x,t)$ in the equation. In both cases, we have to impose additional summability conditions that the coagulation kernel and the initial distribution of particles in order for the evolution equations to make sense. For a thorough survey on the mathematics of the discrete and continuous coagulation equations, and related evolution equations, see [3].

Both of the aforementioned equations assume that there are no new particles fed into the system. However, with an additional *injection* or *source* term, we can encode the

addition of new particles. The injection term then gives the rate at which new particles are injected to the system and this influx of new particles feeds the coagulation process, as the new particles can coalesce with the other particles in the system.

With the additional injection term, the coagulation equations become

$$\begin{aligned} \partial_t n(i,t) = &\frac{1}{2}\sum_{j=1}^{i-1} K(j,i-j)n(j,t)n(i-j,t) \\ &- \sum_{j=1}^{\infty} K(i,j)n(i,t)n(j,t) + s_i, \end{aligned} \tag{1.3}$$

in the discrete case. On the other hand, if we want to allow for a continuum of particle sizes, we arrive at the continuous coagulation equation, which is given by

$$\begin{aligned} \partial_t f(x,t) = &\frac{1}{2}\int_0^x K(x-y,y)f(x-y,t)f(y,t)\,\mathrm{d}y \\ &- \int_0^{\infty} K(x,y)f(x,t)f(y,t)\,\mathrm{d}y + \eta(x). \end{aligned} \tag{1.4}$$

In equation (1.3), the coefficient s_i describes the rate at which new particles of size $i \in \mathbb{N}$ are added to the system. Analogously, in the continuous version (1.4), the function[1] η is such that the value $\eta(x)$ describes the rate of which particles of size $x \in (0, +\infty)$ are added to the system.

In this thesis, we are interested in coagulation equations with injection. In particular, we will try to find out whether there exist stationary solutions to regularized coagulation equations with injection. It will be shown that for certain class of coagulation rate kernels K, there exists[2] a solution to a regularized version of the coagulation equation with injection. The central theorem presented in this thesis was first proved by Ferreira et al. in [6]. The present author was involved in supplementing the necessary details to make the argument rigorous. For this reason, the thesis will follow the structure of the proof presented therein quite closely.

In the next chapter, we flesh out the physical motivation and aspects of coagulation processes, with and without injection. The behavior of coagulation processes related to

[1] The precise formulation of the continuous case needs some work, as we want to be able to describe the cases where η is a pure measure rather than a density function of a measure. We will address the problem of the formulation more closely in the subsequent chapters.

[2] Uniqueness, however, is not guaranteed.

the former case have been extensively studied. We will mostly focus on the mathematical relevance of the problem in the context of atmospheric sciences, most notably aerosol physics. We will also discuss the physical relevance of concept of a stationary solution to the coagulation equation.

The third chapter contains the main mathematical tools, definitions, lemmas and theorems needed in the proof. A reader familiar with point set topology, real analysis and functional analysis can skip this chapter and refer back to it, if need be. We will discuss the rudiments of topological vector spaces and Banach spaces, the concept of the derivative of a vector valued function and the weak*-topology of a dual of a topological vector space. Additionally, we outline parts of the theory of Radon measures and two theorems, both going by the name of the Riesz-Markov-Kakutani representation theorem. The representation theorems enable us to use functional analytic methods in the study of coagulation equations. We will recall two fixed point theorems, namely Banach's fixed point theorem and Schauder's fixed point theorem.

With the mathematical concepts and results at our disposal, it is possible to give a precise definition of the theorem that we aim to prove. This will be the subject of chapter 4. Therein, we will reformulate coagulation equations in a suitable way. We will then give a formulation of the main theorem, and sketch the steps of the subsequent proof. Finally, in chapter 5, we follow these steps and prove the theorem. In the final chapter of the thesis, we mention several problems and theorems related to the topic of this thesis. Most importantly, referring to [6] for futher details, we will describe a classification of coagulation kernels that behave asymptotically in a certain way. This classification provides a criterion for deciding whether a coagulation kernel from this class has a stationary solution or not.

2 Coagulation processes

In this section, we will describe the physical setting behind coagulation equations, the heuristic ideas giving rise to the coagulation equations and motivations for studying their solutions. We will begin by giving an account of the processes that the coagulation equations are supposed to model, namely *coagulation processes*. Afterwards, we elucidate the connection between the coagulation processes and equations and give futher reasons for the specific form of the equations.

Coagulation processes are found in several distinct areas of physics and chemistry, including aerosol physics [8], polymer chemistry [34] and astrophysics [5]. Since coagulation equations are so ubiquitous, we will not give a full account of all possible applications. Rather, we will focus on one particularly important area where coagulation processes are studied, namely aerosol physics. Aerosol physics also serves to motivate coagulation equations with injection, and we will give an interpretation for this new family of coagulation equations. Lastly, we will give a physical account of stationary solutions to coagulation equations.

2.1 Coagulation processes and equations

Consider a system of particles[3] moving about in a volume. We assume that the particles are susceptible of sticking together to form larger particles, should they collide. In other words, we assume that the particles *coalesce* upon collisions. Furthermore, it is assumed that the mass of the particles is conserved as they coalesce. This means that the size of the of the greater particle agrees with the sum of the sizes of the smaller particles that coalesced to form it. Figure 1 illustrates the preservation of total mass as two particles coalesce.

At each point in time $t \geq 0$, we can associate the aforementioned system of particles with a particle size distribution $n(\cdot, t)$, which tells us the size profile of the system. For example, we can take $n(i, t)$ to be the density of particles of $i \in \mathbb{N}$ (how many particles of this size are there per unit of volume) at time $t \in \mathbb{R}_+$. As the coagulation process evolves,

[3]Also known as clusters. One can think of these as clusters of smaller, "atomic" particles, which we will sometimes call monomers. Depending on the context, the monomers can be molecules, and the particles consisting of multiple monomers can be polymers, aerosol particles, or any other conglomerates of smaller particles.

small particles become parts of larger particles and correspondingly the size distribution changes in time.

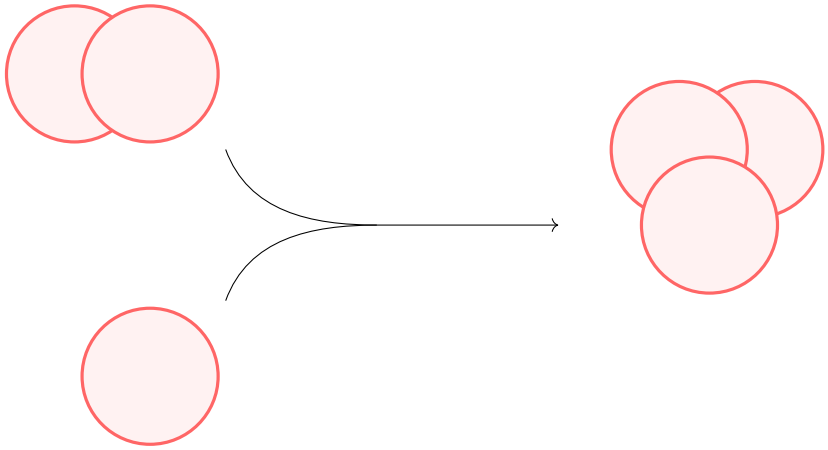

Figure 1: A particle of size 2 coalesces with a particle of size 1 to form a particle of size 3

It could in principle be the case that, as the process evolves, more than two particles collide and coalesce simultaneously. However, in the study of coagulation processes, it is usually assumed that coagulation happens only between two particles. The heuristic reason for this assumption is that since the collision of two particles is already quite rare, the simultaneous collision of three or more particles should be *extremely rare*. This drastically simplifies the mathematical models of coagulation processes, since it allows us to consider how the size distribution evolves as a function of pairwise collisions.

It should be noted that in physical coagulation processes, there is usually a positive chance of two colliding particles not forming a larger particle. Depending on the situation, it is usually assumed that non-sticking collisions are very rare. For this reason, we follow the typical convention and assume that the coagulation probability upon collision is one [12], [24], [9]. These assumption will then be encoded into the equation modeling the process.

Suppose now that two particles of masses/sizes m_1 and m_2, respectively, collide and, for the aforementioned reason, coalesce. The assumption that the total mass is conserved in coalescence then means that the newly formed particle is of mass/size $m_1 + m_2$. This coalescence will be reflected in the size distribution. Indeed, we should see a decrease in densities $n(m_1, t)$ and $n(m_2, t)$ after a small time interval. On the other hand, there should be a slight increase in the density $n(m_1 + m_2, t)$.

We would like to formulate an evolution equation that describes the evolution of the particle size distribution $n(\cdot, t)$ as time evolves. Consider, for the moment, that the particles take only discrete sizes. The change in the time derivative of $n(i, t)$ is determined by two terms. It is increased by smaller particles that come together to form a larger particle of size i. This happens in case particles of size j and size $i - j$ collide, with the masses j and $i - j$ smaller than i. If we assume that the coagulation rate between particles depend only on their sizes (we'll come back to this point later), then the increase in the amount $n(i, t)$ coming from coagulations of the type $(j, i - j)$ should depend multiplicatively on the amounts $n(j, t)$ and $n(i - j, t)$ and the coagulation rate coefficient $K(j, i - j)$. The reasoning behind this is the following. For any given particle of size j, the rate of coalescence with particles of size $i - j$ is proportional to how many possible coagulation pairs there are. In other words, it is proportional to $n(i - j, t)$, and the constant describing the proportionality is given by a number which we denote by $K(j, i - j)$ [1]. The increase given by conglomerations of the type $(j, i - j)$ is therefore $K(j, i - j)n(j, t)n(i - j, t)$ If we sum over all possible pairs and take double-counting into account by introducing the factor $1/2$ in front of the sum, we see that the total increase in $n(i, t)$ is then given by

$$\frac{1}{2}\sum_{j=1}^{i-1} K(j, i - j)n(j, t)n(i - j, t).$$

Similarly, the decrease in density $n(i, t)$ results from particles of size i coalescing with particles of any other size $j = 1, 2, \ldots$, to produce one larger particle of size $i + j$. Therefore, the decrease in the density of particles of size i is given by

$$\sum_{j=1}^{\infty} K(i, j)n(i, t)n(j, t).$$

Combining these two terms, we arrive at *Smoluchowski's coagulation equation*,

$$\partial_t n(i, t) = \sum_{j=1}^{i-1} K(j, i - j)n(j, t)n(i - j, t) - \sum_{j=1}^{\infty} K(i, j)n(i, t)n(j, t).$$

Since $n(i, t)$ is assumed to be the density of particles of size i at time t, then in the three-dimensional case, it has the dimension $\frac{1}{\mathrm{m}^3}$. Therefore, the time-derivative $\partial_t n(i, t)$ has the dimension $\frac{1}{\mathrm{m}^3 s}$. It follows from these considerations that the rate kernel is of the dimension $\frac{\mathrm{m}^3}{s}$.

In physical application, the cluster sizes should always form a discrete subset of the positive real numbers. Thereby, for such applications, the discrete coagulation equation is fundamental. On the other hand, by considering regimes of physically relevant particles with ever finer monomer sizes, we arrive at the continuous coagulation equation

$$\partial_t f(x,t) = \int_{(0,x)} K(y, x-y) f(x,t) f(y,t) \,\mathrm{d}y - \int_{\mathbb{R}_*} K(x,y) f(x,t) f(y,t) \,\mathrm{d}y.$$

This version of the coagulation equation has been studied at least since the 1950s', for example by Melzak [19]. In what follows, we will consider the continuous case with certain imporant modifications, since with a suitable care the discrete case can be shown to reduce to it mathematically.

Up to this point, we have taken the coagulation kernel K as a given. For certain families of K, the solutions to the corresponding coagulation equations are straightforward to analyse and in some cases the solutions even allow for analytical expressions. For example, constant kernels like $K \equiv 1$, addive kernels like $x + y$ and multiplicative kernels like xy [see 18] have solutions that can be represented in terms of the Laplace transform [20]. Solutions for a few simple kernels are presented in [4]. Yet, finding the right kernel to model the coagulation process under consideration is not a trivial task. Indeed, the kernel is assumed to encode most of the relevant information about the system.

In the original work by Smoluchowski [29], he considered a system of spherical particles moving about in the three dimensional space $\mathbb{R}^3$ and surrounded by small, non-coalescing particles. Smoluchowski argued that each particle in the system is essentially moving according to Brownian motion due to collisions with small, non-coalescing particles, and the variance of the Brownian motion of each particle is inversely proportional to its radius $r \sim x^3$. Then, heuristically speaking, the pace at which two particles in the system meet should be proportional to the expression $(r_1 + r_2)/(1/r_1 + 1/r_2)$. Since the particles are assumed to be spherical, this suggests a coagulation rate kernel of the form

$$K(x,y) = C(x^{1/3} + y^{1/3})(x^{-1/3} + y^{-1/3}),$$

where C is some positive constant [21]. We will come back to this kernel in the last section.

Smoluchowski's derivation of the coagulation rate kernel was heuristic, and in general it remains an open problem to try to connect the microscopic description of the system of colliding particles with the macroscopic coagulation equation. In particular, showing

mathematically that a certain coagulation kernel arises from the coagulation dynamics of a microscopic physical systems has not been done as of now. For recent work in this area, see the articles by Hammond and Rezakhanlou [11], [10].

2.2 Coagulation processes in aerosol physics

Aerosol physics is a field where the physical properties of aerosols are studied. Aerosols are solid or liquid particles suspended in gas, for example in the air of the atmosphere [8]. Different types of aerosols include, among others, mist, dust, fog and haze, illustrating just how common various aerosol related phenomena are. Aerosol particles are typically small, with their sizes ranging from nanometers up to several micrometers [9, 23]. They can be formed for example when liquids of solids are broken down into small particles, when fine particles are transported and become suspended in gas, or when small monomer particles congolomerate in gas and form larger particles [9, p. 1].

One possible mechanism by which aerosol particles are formed and enlargened is via coagulation of smaller particles [14]. For example, sulfuric acid and ammonia form larger clusters in this way [22]. In aerosol physics, one typically models the evolution of particle sizes using an equation that takes into account the effect of particles fragmenting to smaller particles, the removal/addition of particles from/to the system [see 31]. These terms, along with diffusion and growth terms, are included in what is known as the General Dynamic Equation [8, p. 307], which is a more complex time-evolution model for the particle size distribution. In what follows, we only consider the coagulation and injection terms. We refer the reader to [6] for the justification of this restriction.

Figure 2 illustrates the evolution of the particle size distribution. It is based on the data collected at the SMEAR III measurement station, which is located in Helsinki, Finland [25]. Each vertical slice in the picture is a measurement of the size distribution profile at a given time. Note that the distribution varies from time to time, but in the scale of minutes it stays approximately stationary.

The physical properties of aerosol particles depend heavily on their sizes, which is one of the reasons for studying the aerosol size distribution both empirically and using models. Indeed, one of the central aims of the study of aerosols is to model and measure the densities of particles belonging to certain size ranges in the system under study. Understanding the size distribution is also practically important. For example, when aerosol particles reach certain sizes, they can act as cloud condensation nuclei (CCN) [13]. Cloud

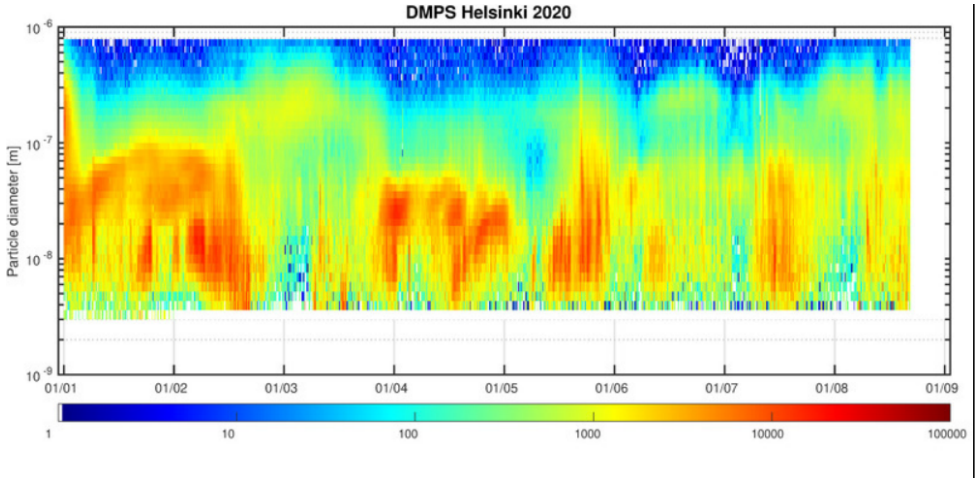

Figure 2: Evolution of aerosol particle size distribution in Kumpula, Helsinki during the beginning of January 2020. The warmer a point (x, y) is, the larger the concentration of particles of size y at time x. The picture is based on measurements taken at the SMEAR III station. The picture (cropped from the original one) is taken from https://wiki.helsinki.fi/display/SMEAR/DMPS, where up to date data can be found. For more information on these and other measurements, see [25].

condensation nucleid are particles on which water can start to vaporise, and they constitute a crucial factor in cloud formation. Consequently, understading how aerosol particle size distributions behave has ramifications on the understanding of climate and the development of atmospheric models [15, p. 144]. In addition, aerosols have various negative effects on human health [23].

2.3 The injective term

In previous description of coagulation processes, we considered systems where all the particles in the system have been there from the very beginning and the total mass of the particles is preserved throughout the evolution. Once a larger particle is formed from two smaller ones, it does not break up, so the average size of the particles will grow without restrictions. For example, once all monomer particles have coalesced with other particles, there are no monomers left in the system. In the previous section, we briefly mentioned that in some coagulation processes, new particles can be injected into the system during

the time-evolution. We might add new monomers, both monomers and larger particles, or just larger particles. In all of these cases, we well call the addition of new particles *injection*. How many new particles are added at each moment in time might vary, but we will mostly be interested in the case where the injection does not change in time.

The injection of new particles into the system changes the evolution of the particle size distribution drastically. If no injection is present and if there is a lower bound on the size of the particles in the system, all small particles eventually vanish as they become parts of larger particles. On the other hand, adding enough small particles into the system might prevent the density of small particles from vanishing entirely. Moreover, it is not immediately clear whether coagulation systems with injection have stationary solutions. We will adress this question in this thesis.

If we consider the coagulation process where aerosols coalesce in a certain volume in the atmosphere, then the injection term accounts new coalescing particles fed into the system. These new particles can come from various different sources in the surrounding environment. The amount of new particles added to the system varies in time, but in general it should stay approximately constant even in time scales that are many orders of magnitude larger than the time scale of individual particle collisions. Therefore, we will model these processes by encoding the added particles to the aforementioned constant injection term. We then modify the coagulation equation by incorporating the injection of new particles as a term in the equation. This is the underlying idea behind the following equation, which we call the coagulation equation with injection. Here η denotes the injection term.

$$\begin{aligned} \partial_t f(x,t) = &\int_{(0,x)} K(y, x-y) f(x,t) f(y,t) \, \mathrm{d}y - \int_{\mathbb{R}_*} K(x,y) f(x,t) f(y,t) \, \mathrm{d}y \\ &+ \eta(x) \end{aligned} \tag{2.1}$$

Note that for this equation to make sense, we are for the moment assuming that $f(\cdot, t)$ and η both have continuous densities. We will lax this assumption in what follows, but Equation (2.1) should give an intuitive idea how the injection term affects the time-evolution of f.

2.4 Stationary solutions

Informally, a solution to the coagulation equation is stationary, in case the size distribution profile stays constant as the system evolves. If there is an injection term present, the rate

at which new particles are added must be in balance with the escape of the sizes of the coalescing particles towards infinity. Provided that there exists such a balanced state, then the size-distribution does not change even though particles are constantly coalescing. We call stationary solutions to the coagulation equation those particle size distribution functions that do not change under the evolution described by the coagulation equation with injection, i.e. whose time-derivative is zero. The rate at which added particles flow towards larger particle sizes in stationary solutions is derived in [6].

Finding stationary solutions to the coagulation equations with injection is of course a theoretically interesting problem. Do the coagulation equations even posess stationary solutions, and can we give certain conditions that guarantee the existence of these solutions? Additionally, there is also a practical side to the story. Recall that in Figure 2, we can see approximately stationary states in the size distributions in the scale of minutes. If we can show that coagulation equations with injection have stationary solutions, and if other solutions evolve towards these stationary solutions, this might help us understand why we can see nearly stationary states in the particle size distribution data.

In Section 5, we will show that for a suitably regularized coagulation equation with injection, we can find stationary solutions. While this does not directly answer the questions posed in the above paragraph, it is a central result in a proof that certain coagulation kernel admit stationary solutions [6]. Therefore, with the correct mathematical formulation, the answer to the first question is partly affirmative. We will discuss this in more detail in Section 6, where we will also give conditions on the existence and non-existence of stationary solutions.

3 Mathematical preliminaries

In this section, we go through the mathematical concepts needed to understand the proof at hand, namely the proof of the existence of nontrivial stationary solutions to the regularized coagulation equations with suitably picked kernels. The proof relies heavily on existing mathematical results and techniques. Among the most important tools are duality techniques and the theory of Radon measures, fixed point techniques, and Fréchet derivative techniques. Thus, we need to cover some results in functional and real analysis, in topology, and in fixed point theory. Anyone familiar with functional and advanced real analysis should be able to skip this chapter.

Even in case the reader is not familiar with the following results, we will still assume that they are familiar with the basics of measure theory and topology. We will first define the concept of a topological vector space and go through various relevant topological concepts and results. After this, we will move on to treat the theory of Banach spaces, which constitute an important class of topological vector spaces. We will then formulate two fixed point theorems needed in this thesis, namely those of Banach and Schauder. Proceeding to real analysis and measure theory, we define Radon measures and formulate the representation theorem of Riesz, Markov and Kakutani, which gives us a way of identifying measures with continuous linear functionals on $C_0(X)$[4]. We also state some results concerning the decomposition of signed measures to positive and negative parts.

3.1 Topological Vector Spaces

Topological vector spaces form the structural basis on which we build our set of tools. They unify the concepts of topological space and vector space in a consistent way. We assume that the reader knows the definitions of a topological space and a vector space, but we still make a few remarks about the topological spaces we want to consider.

As a note on terminology, in this thesis we use the term *neighborhood* of a point to stand for something that is sometimes known as an *open neighborhood*. In other words, according to our terminology, in a topological space (X, τ_X) a neighborhood of a point $x \in X$ is a set $U \in \tau_X$ such that $x \in U$.

To formulate and prove the results in this thesis, we need some concepts of topology,

[4]Recall that $C_0(X)$ is the sup-norm closure of the space $C_c(X)$. Appendix A recounts some notation used in the thesis.

including Hausdorff spaces and compact topological spaces.

Definition 3.1 (Hausdorff space)**.** A topological space (X, τ_X) satisfies the *Hausdorff separation property*, in case every two disjoint points $x, y \in X$ have some disjoint neighborhoods. In other words, if there exist neighborhoods $U_x, U_y \in \tau_X$ such that $U_x \cap U_y = \emptyset$. If a space satisfies this property, we call it a *Hausdorff space*. ◁

A typical example of a Hausdorff space is the set of real numbers $\mathbb{R}$ equipped with the topological structure coming from absolute value metric on $\mathbb{R}$. This is also a metric space.

Definition 3.2 (Metrisable topological space)**.** We say that a topological space (X, τ_X) is metrisable, if there exists some metric $d\colon X \times X \to \mathbb{R}$ such that the topology induced by d, denoted τ_d, matches τ_X, i.e. $\tau_d = \tau_X$. ◁

Definition 3.3 (Compact space)**.** A topological space (X, τ_X) is called *compact*, if every open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ has a finite subcover.

Remark. Here $\mathcal{A}$ is an arbitrary index set, in particular, it might be uncountable.

◁

In this thesis, we are mostly interested in compact subsets of a larger topological space. Recall that subset of a space is called a *compact set*, in case it is compact in the relative topology inherited from the larger topological space. Equivalently, a subset of a topological space is compact if every open cover of the subset has a finite subcover.

The following concept of sequential compactness, which in general is distinct from compactness, is equivalent to compactness in certain special spaces. In particular, the two concepts imply one another in metrisable spaces. Clearly this equivalence is inherited by all subsets of metrisable spaces, since subset of metrisable spaces are also metrisable when equipped with the relative topology.

Definition 3.4 (Sequentially compact space)**.** A topological space (X, τ_X) is sequentially compact, if every sequence $(x_i)_{i=1}^\infty$ has a converging subsequence $(x_{i_k})_{k=1}^\infty$. Here i_k is increasing. ◁

Proposition 3.5 (Equivalence of compactness and sequential compactness in metrisable spaces)**.** *Suppose that (X, τ_X) is a metrisable topology. Then it is X is compact if and only if it is sequentially compact.*

Recall that if $C \subset X$ where X is a topological space, we define the closure of C, denoted by $\overline{C}$, to be the collection of those points whose every neighborhood meets C.

Definition 3.6 (Separable space)**.** We call a topological space (X, τ_X) *separable*, in case it contains a countable subset $C \subset X$ that is dense in X, i.e. $\overline{C} = X$. $\triangleleft$

We also need the concept of local compactness, which, together with the Hausdorff-property, plays a role in the Riesz-Markov-Kakutani theorem.

Definition 3.7. A topological space (X, τ_X) is *locally compact*, in case every point $x \in X$ has a neighborhood $U \in \tau_X$ such that the closure $\overline{U}$ is compact. $\triangleleft$

Note that the following simple fact concerning local compactness holds.

Fact 3.8. *The space $\mathbb{R}_* \coloneqq (0, +\infty)$ endowed with the relative topology inherited from the space $\mathbb{R}$ with the usual topology is a locally compact Hausdorff space.*

Having now discussed some topological properties that a topological space might possess, we turn to the concept of a topological vector space. Since we are discussing vector spaces, which must be defined over a scalar field, we shall adopt $\mathbb{K}$ to stand for either the real scalar field $\mathbb{R}$ or the complex scalar field $\mathbb{C}$. These both have natural topologies, which are induced by, for example, the absolute value in case of $\mathbb{R}$ and the modulus in case of $\mathbb{C}$. In this thesis, we only need the real field $\mathbb{R}$.

We will use the following definition of a topological vector space, which can be found in Rudin's textbook [27, p. 7].

Definition 3.9 (Topological Vector Space)**.** Consider a topological space (X, τ_X), where X is also a vector space over the field $\mathbb{K}$. We call X a *topological vector space*, sometimes more concisely a TVS, in case the following conditions are met. First of all, the vector space operations are compatible with the topological structure, which means that the addition

$$\begin{aligned} X \times X &\to X \\ (x, y) &\mapsto x + y \end{aligned}$$

and the scalar multiplication maps

$$\begin{aligned} \mathbb{K} \times X &\to X \\ (\alpha, x) &\mapsto \alpha x \end{aligned}$$

should all be continuous with respect to the obvious topologies. Secondly, every singleton set $\{x\}$, where $x \in X$, must be closed. $\triangleleft$

The assumption that all singleton sets $\{x\}$, where $x \in X$, are closed in the topology of X is explicitly stated in the definition of a topological vector space. Recall that this is one of the key separation properties in topological spaces. In fact, we have another, stronger separation condition. Namely, it turns out that the assumptions of a topological vector space imply that X is a Hausdorff space, which means that two distinct points in X have disjoint neighborhoods.

Proposition 3.10. *Suppose that (X, τ) is a topological vector space according to Definition 3.9. Then (X, τ) is also a Hausdorff topological space.*

Proof. See any book on topological vector spaces. For example [27]. $\square$

Finally, we will define the concept of the topological dual of a topological vector space.

Definition 3.11 (Dual space of a topological vector space)**.** Let (X, τ_X) be a topological vector space. The topology on X and the standard topology on the field $\mathbb{K}$ (which can be $\mathbb{R}$ or $\mathbb{C}$, depending on the respective field of X as a vector space), make a collection of linear maps $\alpha\colon X \to \mathbb{K}$ continuous. The collection of these continuous linear maps $X \to \mathbb{K}$ is denoted by X^*. The collection X^* is called the topological dual space of X. $\triangleleft$

3.2 Banach spaces

Vector spaces allow us to multiply elements by a scalar and to add two elements together. In many cases, we additionally want to have the tools to gauge the length of an element in the vector space, just like we can assign a length to a vector in the prototypical vector space $\mathbb{R}^d$ with the euclidean structure. Spaces where we can gauge the length of any vector are called *normed space*. We assume that the reader is familiar with this concept and the properties of the norm in the normed space. Otherwise, the definition and basic properties of normed spaces are discussed in [27, pp. 3–5]. The norm induces a metric which induces a topology on the normed space.

In normed spaces, which are metric spaces, it makes sense to define the notion of a Cauchy sequence, which is a sequence whose elements become arbitrarily close as we go on in the sequence. In many cases, it is desirable that whenever we have a Cauchy sequence, there

exists a limit of that sequence in our space. When this property is satisfied, we say that the space is *complete*.

Definition 3.12 (Banach space)**.** A normed space $(X, \|\cdot\|_X)$, where the vector space X might be infinite dimensional, is called a *Banach space*, in case it is complete, i.e. if every Cauchy sequence convergences to a point in X. ◁

We are mostly interested in linear maps between Banach spaces. The normed structure of Banach spaces gives us a way to characterise the behaviour of linear maps between spaces.

Definition 3.13 (Boundedness of an operator)**.** Let X, Y be two Banach spaces. We say that a linear map (in other words, a linear operator) $T\colon X \to Y$ is bounded, in case there exists a constant $C \geq 0$, such that

$$\|Tx\|_Y \leq C\|x\|_X$$

for every $x \in X$. We call smallest constant C that satisfies the above property the *operator norm* of T, and denote it by $\|T\|$. Consequently, $\|Tx\| \leq \|T\|\|x\|$. ◁

In Banach spaces, the continuity of an operator is closely tied with the boundedness of that operator. More precisely, the following proposition is true.

Proposition 3.14 (Boundedness and continuity)**.** *Let X and Y be Banach spaces. Then a linear mapping $T\colon X \to Y$ is bounded in the sense of Definition 3.13, if and only if T is continuous. Both conditions are also equivalent to T being continuous at any given point $x \in X$.*

Proof. See any book on elementary functional analysis or operator theory, like [2]. □

The following theorem allows us to extend continuous linear operators from dense sets to the whole space.

Proposition 3.15. *Let X, Y be Banach spaces and suppose that $T\colon S \to Y$ is a continuous linear operator from a dense subset $S \subset X$. Then, there exists a unique bounded/continuous extension of T, denoted here by $\hat{T}\colon X \to Y$, which is a linear and bounded operator with $\|\hat{T}\| = \|T\|$.*

Finally, we note that the collection of continuous mappings from an interval to a Banach space forms a new Banach space with the supremum norm. In particular, the following result is true.

Proposition 3.16. *Suppose that Y is a Banach space. Then for any real number $T > 0$, the space of continuous functions $C([0,T], Y)$ equipped with the norm*

$$\|f\|_\infty := \sup_{t\in[0,T]} \|f(t)\|$$

is a Banach space.

3.3 Fixed Point Theorems

In many situations, like when solving differential or integral equations, one is interested in finding points in the space of solutions that are invariant with respect to an application of some map f, i.e. finding elements x that satisfy $x = f(x)$. Suggestively, such points are called fixed points. Fixed point theorems give us assumptions on the space X and the mapping f which guarantee that there exists a fixed point. Of course, both the space and the mapping must be suitably well-behaved for fixed points to exists.

Banach's fixed point theorem is a very powerful tool, which allows us to find *unique* fixed points. Additionally, the proof also gives an iterative procedure for approximating the fixed point together with an estimate on the rate of convergence. Recall that a contraction mapping on a metric space X is a mapping $f\colon X \to X$, which satisfies $d_X(f(x), f(y)) \leq k d_X(x, y)$ for all $x, y \in X$, where k is some *contraction constant* $k \in [0, 1)$.

Theorem 3.17 (Banach's fixed point theorem [see 28, p. 470])**.** *Let (X, d) be a complete metric space, and let $K\colon X \to X$ be a contraction mapping on X with a contraction constant $k \in [0, 1)$. Then there exists a fixed point $x^* \in X$. That is, $Kx^* = x^*$. Moreover, this fixed point is necessarily* unique*.*

Proof. See, for example, the proof by Simon in [28, p. 470]. □

Recall the fundamental theorem of Brouwer, which states that every continuous function from a compact subset of the real numbers to itself has at least one fixed point. We take this theorem and its generalizations to higher dimensions as given.

Theorem 3.18 (Brouwer's fixed point theorem)**.** *Let $d \in \mathbb{N}$ and suppose that the subset $K \subset \mathbb{R}^d$ is non-empty, convex and compact. Then any continuous map $f\colon K \to K$ has a fixed point $x^* \in K$.*

Proof. One proof can be found in Zeidler's textbook [33, p. 51]. □

Remark. The fixed point is not necessarily unique. As simple example, consider the mapping $f\colon [0,1] \to [0,1]$ with $f(x) = x$. This mapping satisfies the hypotheses of Brouwer's fixed point theorem and has uncountably many fixed points.

Note that in Brouwer's fixed point theorem, we are constrained to studying functions on the euclidean spaces $\mathbb{R}^n$. In the subsequent proof, this result is simply not applicable. Fortunately, there is a stronger fixed point theorem. This is known as *Schauder's fixed point theorem* . Schauder's fixed point theorem guarantees that a continuous map $T\colon K \to K$ from a subset of a Banach space X to itself has a fixed point, provide that the domain set K satisfies certain conditions.

Theorem 3.19 (Schauder's fixed point theorem)**.** *Consider a Banach space X with a compact, convex and non-empty subset $K \subset X$. If an operator $T\colon K \to K$ is continuous with respect to the topology of X, then there exists a point $x^* \in K$ such that $Tx^* = x^*$, i.e. x^* is a fixed point of the operator T.*

Proof. For the above formulation and a corresponding proof of this theorem, see for example [33, p. 56] or [2]. The proof essentially relies on Brouwer's fixed point theorem. □

In the proof of Schauder's fixed point theorem, Brouwer's fixed point theorem is used in the construction of the eventual fixed point. Therefore, it should be noted that Schauder's fixed point theorem gives no guarantee of the uniqueness of the mentioned fixed point. Indeed, there can be many, even infinitely many, fixed points in a situation satisfying assumptions. When Schauder's fixed point theorem is applied to study partial differential equations, this inherent non-uniqueness of fixed points might be reflected in the inability to show that a certain problem has a unique solution. We will run into this shortcoming later on in this thesis, and we will only be able to show the existence, not uniqueness, of solutions to a certain problem.

3.4 Theory of Radon measures

We assume that the reader has come across the concept of a σ-algebra, and the notions of a measurable space and a measure on such a space. In this thesis, we are interested in measures with respect to which all Borel sets are measurable. From this, the notion of Borel measure arises.

Definition 3.20 (Borel measure)**.** Let (X, τ_X) be a topological space. Denote by $\mathcal{B}(X)$ the σ-algebra generated by open set in this topological space. Now $(X, \mathcal{B}(X))$ constitutes a measurable space. Let $\mu\colon \mathcal{B}(X) \to [0, +\infty]$ is a measure on $(X, \mathcal{B})$. In this case, we call μ a Borel measure on X, since every Borel set is μ-measurable. ◁

When we have a Borel measure, it makes sense to ask whether the measure approximates certain sets well. To this end, we define what it means for a Borel measure to be *inner* or *outer* regular. For general theory of Borel measures, see [17] and [26], where the following definitions also appear.

Definition 3.21 (Inner regular measure)**.** We call a Borel measure μ in a measure space $(X, \mathcal{A}, \mu)$ an *inner regular measure*, in case for every U open in X, we have

$$\mu(U) = \sup\{\mu(K)\colon K \subset U, K \text{ is compact}\}$$

◁

Definition 3.22 (Outer regular measure)**.** We call a Borel measure μ in a measure space $(X, \mathcal{A}, \mu)$ an *outer regular measure*, in case for every Borel set $U \in \mathcal{B}(X)$, we have

$$\mu(U) = \inf\{\mu(V)\colon U \subset V, V \text{ is open}\}$$

◁

If we combine these conditions, with the additional requirement that every compact set is of finite measure, we get an important class of measures, namely the positive Radon measures.

Definition 3.23 (Positive Radon measure)**.** Let (X, τ_X) be a locally compact Hausdorff space. We call a Borel measure μ in a measure space $(X, \mathcal{A}, \mu)$ a *positive Radon measure*, in case μ is both inner and outer regular, and if for every compact $K \subset X$, we have $\mu(K) < \infty$. ◁

Definition 3.24 (Bounded measure)**.** A measure μ on a measurable space $(X, \mathcal{A})$ is bounded, in case $\mu(X) < +\infty$. ◁

For example, the collection of positive Radon measures includes all n-dimensional Lebesgue measures on $\mathbb{R}^n$, the Dirac delta measures based on point $a \in \mathbb{R}$, denoted by δ_a, and positive linear combinations of these measures. The Dirac delta measures are also bounded Radon measures, while the Lebesgue measures are not. We denote the set of all positive Radon measures on X by $\mathcal{M}_+(X)$ and the collection of all bounded positive Radon measures on X by $\mathcal{M}_{+,b}(X)$.

One of the most important results that we need in this thesis is the fact that we can in some sense identify positive Radon measures on a locally compact Hausdorff space with the positive functionals $C_c(X) \to \mathbb{R}$. To be more precise, the following statement holds:

Theorem 3.25. *Suppose X is a topological space that is both locally compact and Hausdorff. Let $\phi\colon C_c(X) \to \mathbb{R}$ be a linear functional satisfying the additional property that $\phi(f) \geq 0$ for every positive $f \in C_c(X)$. Then there exists a unique* Radon *measure on X such that for every $f \in C_c(X)$*

$$\phi(f) = \int_X f(x)\,\mathrm{d}\mu(x).$$

Moreover, the Radon measure has the additional property that it is inner regular also on sets of finite measure, not only on open sets.

Proof. For two slightly different statements and proofs of this theorem, corresponding to slightly different definitions of Borel (and thus Radon) measures, see Rudin [26, p. 40] and Folland [7, p. 212]. The proofs are straightforward in both cases, and require knowledge of Uhrysohn's lemma, partitions of unity and basic measure theory. Our statements aligns more closely with Folland's version. □

We will sometimes mention in passing the support of a Radon measure.

Definition 3.26 (Support of a Radon measure)**.** Let (X, τ_X) be a topological space and let μ be a Radon measure defined on this space. We define its support $\operatorname{supp}(\mu)$ as the closure of the set of all those points $x \in X$ such that every neighborhood of x has a positive measure. ◁

The previous discussion concerns positive measures, which assing nonnegative values to measurable sets of the space. Signed measures, on the other hand, assign real values to measurable sets.

Definition 3.27 (Signed measure)**.** Given a measurable space $(X, \mathcal{A})$, we say that a map $\mu\colon \mathcal{A} \to \mathbb{R}$ is a signed measure, when for every $A \in \mathcal{A}$, we have

$$\mu(A) \coloneqq \sum_{i=1}^{\infty} \mu(A_i), \tag{3.1}$$

where $\{A_i\}_i$ with $A_i \in \mathcal{A}$ is a partition of A.

Remark. This is not the only commonly used definition of signed measure. Sometimes, only one of the positive and negative parts of the sum in (3.1) is required to be finite in the definition of signed measures. The upshot of our definition is that signed measures are always bounded, i.e. all measurable sets are of finite measure.

◁

Since all rearrangements in (3.1) are equal by assignment, the series in the definition converges absolutely. Moreover, the values are finite, so $\mu(A) \in \mathbb{R}$ for every $A \in \mathcal{A}$.

In order to have some examples and counter-examples in mind, recall that Dirac delta-measures centered at points in $\mathbb{R}_*$, the restriction of the 1-dimensional Lebesgue measure[5] on compact subsets $K \subset \mathbb{R}_*$, and their linear combinations are all signed Radon measures on $\mathbb{R}_*$ according to the above definition. On the other hand, the restriction of the Lebesgue measure on $\mathbb{R}_*$ is not a signed measure according to this definition, since the total measure of the space $\mathbb{R}_*$ is not bounded.

Proposition 3.28 (Decomposition of signed measures)**.** *Suppose that*

$$\mu\colon \mathcal{A} \to \mathbb{R}$$

is a (finite) signed measure on a measurable space $(X, \mathcal{A})$. Then there exists a unique decomposition of μ into two bounded positive measures μ_+, μ_- on $(X, \mathcal{A})$ such that $\mu(A) = \mu_+(A) - \mu_-(A)$. For $A \in \mathcal{A}$. We write $\mu = \mu_+ - \mu_-$.

[5] Or its restriction to the σ-algebra of Borel-sets

Proof. This is known as Jordan's decompostion theorem. In Folland's textbook [7], this theorem and the Hahn decomposition theorem, which is presupposed by Jordan's decomposition theorem, are proved for signed measures with possibly infinite values. However, essentially the same proof can be used when dealing with our definition of signed measures. □

The above proposition allows us to extend the definition of Radon measure to signed measures. This gives rise to the concept of signed Radon measure.

Definition 3.29 (Signed Radon measure)**.** A signed measure μ on $(X, \mathcal{A})$ is called a *signed Radon measure*, in case both μ_+ and μ_- in the Jordan decomposition of μ are (positive) Radon measures according to Definition 3.23. ◁

By the Jordan decomposition theorem, each signed measure μ splits into two parts μ_+ and μ_-, which are both positive (bounded) Radon measures. Instead of considering the difference $\mu = \mu_+ - \mu_-$, which yields the original signed measure, we can also consider the sum of the negative and positive parts, which gives rise to a new positive Radon measure. This is how we construct the total variation measure corresponding to a signed Radon measure μ.

Definition 3.30 (Total variation measure)**.** Let μ be a signed measure on a measurable space $(X, \mathcal{A})$, which can be decomposed as $\mu_+ - \mu_-$, with $\mu_+, \mu_- \in \mathcal{M}_{+,b}(X)$. We define the *total variation measure* of $\mu\colon \mathcal{A} \to \mathbb{R}$, typically written as $|\mu|$, by

$$|\mu|(A) = \mu_+(A) + \mu_-(A), \quad A \in \mathcal{A}.$$

In other words, $|\mu| = \mu_+ + \mu_-$.

Remark. Since $\mu_+, \mu_- \in \mathcal{M}_{+,b}(X)$, and this set is a positive cone in the space of bounded Radon measures, clearly $|\mu| \in \mathcal{M}_{+,b}(X)$ as well.

◁

We denote the space of signed Radon measures on X by $\mathcal{M}_b(X)$[6]. It is straightforward to show that $\mathcal{M}(X)$ is a real vector space. Moreover the total variation measure gives us a way of endowing $\mathcal{M}_b(X)$ with a normed space structure

[6]Signed Radon measures, per our definition, are bounded. We put the subscript "b" under $\mathcal{M}$ to stress this.

Proposition 3.31. *The space $\mathcal{M}_b(X)$ of signed Radon measures on a locally compact Hausdorff space X is a normed space, when we define a norm $\|\cdot\|$ on $\mathcal{M}_b(X)$ by*

$$\|\mu\| = |\mu|(X) = \mu_+(X) + \mu_-(X).$$

Moreover, as a topological space, this space is complete. Consequently, the normed space $(\mathcal{M}_b(X), \|\cdot\|)$ constitutes a Banach space.

There is another variation of the Riesz-Markov-Kakutani theorem, which states that there is an isometric correspondence between the singed Radon measures over a topological space X and the bounded (equivalently, continuous) linear operators on the Banach space $C_0(X)$.

Theorem 3.32 (Signed Riesz-Markov-Kakutani)**.** *Assume that (X, τ_X) is a locally compact Hausdorff space, and let $\Lambda\colon C_0(X) \to \mathbb{R}$ be a continuous linear functional on the space $C_0(X)$. Then there exists a signed Radon measure $\mu_\Lambda \in \mathcal{M}_b(X)$, which corresponds to the functional Λ in the following manner: For every function $f \in C_0(X)$, the action of Λ on f is the same as integrating f over the whole space with respect to the measure. In other words,*

$$\Lambda(f) = \int_X f \,\mathrm{d}\mu_\Lambda. \tag{3.2}$$

The correspondence between the functional Λ and the measure μ_Λ also preserves the norm. Namely, if we denote by $\|\cdot\|_{tv}$ the total variation norm on $\mathcal{M}(X)$ and by $\|\cdot\|$ the operator norm on the space of continuous linear functionals $C_0(X) \to \mathbb{R}$, then we have

$$\|\mu_\Lambda\|_{tv} = \|\Lambda\| \tag{3.3}$$

Vice versa, to every signed Radon measure $\mu \in \mathcal{M}_b(X)$, there corresponds a continuous functional $\Lambda_\mu\colon C_0(X) \to \mathbb{R}$ given by

$$\Lambda_\mu(f) = \int_X f \,\mathrm{d}\mu, \quad f \in C_0(X),$$

and

$$\|\mu\|_{tv} = \|\Lambda_\mu\|.$$

Thereby, using this isometric bijection, we can say that the spaces $\mathcal{M}_b(X)$ with the total variation norm and $(C_0(X))^*$ with the operator norm are isometrically isomorphic:

$$\mathcal{M}(X) \cong (C_0(X))^*.$$

We can thus treat these two as one and the same space when only their Banach space structure is considered.

Proof. One particularly clearly written and detailed proof, which assumes the first Riesz-Markov-Kakutani theorem relating positive functionals with positive Radon measures, is given by Rudin in [26, pp. 129–132]. □

In this thesis, we are mostly interested in the case $X = \mathbb{R}_*$, where $\mathbb{R}_*$ is equipped with the standard relative topology. Clearly $\mathbb{R}_*$ is locally compact and Hausdorff. On the other hand, signed Riesz-Markov-Kakutani representation theorem makes the following proposition easy to prove.

Proposition 3.33. *The space of signed Radon measures with the total variation norm $(\mathcal{M}_b(\mathbb{R}_*), \|\cdot\|)$ is a Banach space.*

Proof. Verifying that $\mathcal{M}_b(\mathbb{R}_*)$ is a vector space is straightforward. First, we note that the zero measure is a signed Radon measure. Both the positive and negative Radon measures form pointed cones, so positive combinations of both still lie in their respective sets. We can couple this with the Jordan decomposition theorem to get the vector space property. Similarly straightforward is verifying that the total variation is a norm on this space.

The nontrivial part is in showing that the space is complete with respect to the metric induced by the total variation norm. Let (μ_i) be a Cauchy sequence of signed Radon measures. To each of these, there corresponds a unique bounded linear functional $\Phi_i \colon C_0(\mathbb{R}_*) \to \mathbb{R}$ such that $\|\Phi_i\| = \|\mu_i\|$, where on the left hand side we have the operator norm. Moreover, if Φ_i corresponds to μ_i and Φ_j to μ_j, then $\Phi_i + \Phi_j$ corresponds to the sum measure given by $\mu_i + \mu_j$. Similarly multiplying a measure by a scalar number is reflected in the functional side as a multiplication by the same scalar. These considerations yield that

$$\|\Phi_i - \Phi_j\| = \|\mu_i - \mu_j\|,$$

and so $(\Phi_i)_{i\in\mathbb{N}}$ is a Cauchy sequence. The space $(C_0(\mathbb{R}_*))^*$ is complete with respect to the operator norm, so there exists $\Phi \colon C_0(\mathbb{R}_*) \to \mathbb{R}$ such that $\Phi_i \to \Phi$ in $(C_0(\mathbb{R}_*))^*$.

This Φ is linear and bounded, so by the signed version of the Riesz-Markov-Kakutani representation theorem 3.32, there exists a corresponding unique signed Radon measure $\mu \in \mathcal{M}_b(\mathbb{R}_*)$ satisfying $\|\mu\| = \|\Phi\|$. Now, all we need to do is verify that $\|\mu - \mu_i\| \to 0$ as $i \to \infty$. But this follows, since

$$\|\mu - \mu_i\| = \|\Phi - \Phi_i\| \to 0.$$

Consequently, each Cauchy sequence $(\mu_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_*$ converges to an element in this space. This concludes the proof that $(\mathcal{M}_b(\mathbb{R}_*), \|\cdot\|)$ is a Banach space. □

Fact 3.34. *The set of bounded positive Radon measures,*

$$\mathcal{M}_{+,b}(\mathbb{R}_*) \coloneqq \mathcal{M}_b(\mathbb{R}_*) \cap \mathcal{M}_+(\mathbb{R}_*),$$

is a pointed cone. That is, for every $\mu, \nu \in \mathcal{M}_{+,b}(\mathbb{R}_*)$, and for every $\alpha, \beta \geq 0$, the combination of these measures, $\eta \coloneqq \alpha\mu + \beta\nu$, satisfies $\eta \in \mathcal{M}_{+,b}(\mathbb{R}_*)$. We say that $\mathcal{M}_{+,b}(\mathbb{R}_*)$ forms a positive cone *of measures inside $\mathcal{M}_b(\mathbb{R}_*)$.*

Remark. Similar fact holds when we replace $\mathbb{R}_*$ with any Borel subset of $\mathbb{R}_*$.

Lemma 3.35. *The pointed cone of positive, bounded Radon measures $\mathcal{M}_{+,b}(\mathbb{R}_*)$ is a closed subset of the normed space of signed bounded Radon measures $\mathcal{M}_b(\mathbb{R}_*)$ equipped with the total variation norm $\|\mu\| \coloneqq |\mu|(\mathbb{R}_*)$.*

Proof. We define a map

$$\begin{aligned} \Lambda \colon \mathcal{M}_b(\mathbb{R}_*) &\to \mathbb{R} \\ \eta &\mapsto \|\eta\| - \eta(\mathbb{R}_*) \end{aligned}$$

Here $\|\eta\|$ is the total variation norm of the measure η and $\eta(\mathbb{R}_*)$ is the measure of the whole space. If $\|\eta - \nu\| < \varepsilon$, then

$$|\Lambda(\eta) - \Lambda(\nu)| \leq |\|\nu\| - \|\eta\|| + |\nu(\mathbb{R}_*) - \eta(\mathbb{R}_*)| < 2\varepsilon.$$

Consequently, Λ is continuous map from $(\mathcal{M}_b(\mathbb{R}_*), \|\cdot\|)$ to the real numbers equipped with the standard topology.

Suppose that η is a purely positive measure. By the uniqueness of the decomposition of a measure to positive and negative parts, we have $\eta = \eta_+$, and the negative part η_- is the zero measure. This means that

$$\Lambda(\eta) = \|\eta\| - \eta_+(\mathbb{R}_*) = \eta_+(\mathbb{R}_*) + \eta_-(\mathbb{R}_*) - \eta_+(\mathbb{R}_*) = \eta_-(\mathbb{R}_*) = 0.$$

Suppose, conversely, that η satisfies $\Lambda(\eta) = 0$. It immediately follows that $\eta_+(\mathbb{R}_*) = |\eta|(\mathbb{R}_*)$, and so the negative part of the measure vanishes, $\eta_-(\mathbb{R}_*) = 0$. This means that $\eta = \eta_+$, which is just another way of saying that η is a positive measure.

Together these considerations mean that $\mathcal{M}_{+,b}(\mathbb{R}_*) = \Lambda^{-1}(\{0\})$. The set $\{0\} \subset \mathbb{R}$ is closed in the standard topology. Therefore, by continuity of Λ, its preimage $\mathcal{M}_{+,b}(\mathbb{R}_*)$ is closed in the norm topology on $\mathcal{M}_b(\mathbb{R}_*)$. □

Recall that a subset U of a vector space X is called *convex*, if for every $x, y \in U$, we have $tx + (1-t)y \in U$ for every $t \in [0,1]$.

Corollary 3.36. *The cone of positive, bounded Radon measures $\mathcal{M}_{+,b}(\mathbb{R}_*)$ is a closed, convex subset of the space of signed, bounded Radon measures $\mathcal{M}_b(\mathbb{R}_*)$.*

Proof. The convexity of the cone follows from the fact that positive scalar multiples of positive Radon measures are still positive Radon measures, and multiplying by a constant does not affect the boundedness of a bounded Radon measure. In addition, the sum of two positive, bounded Radon measures is one as well. Therefore, the linear combination

$$t\nu + (1-t)\eta, \quad t \in [0,1]$$

of two positive, bounded Radon measures ν, η is still a positive, bounded Radon measure. This means that $\mathcal{M}_{+,b}(\mathbb{R}_*)$ is convex. It is closed by Lemma 3.35. □

In this thesis, we often use the following result for general bounded Radon measures.

Proposition 3.37. *Let μ be a positive, bounded Radon measure on $\mathbb{R}_*$ and $f \in L^1(\mu)$ an integrable function with respect to this measure. Then,*

$$\left| \int_{\mathbb{R}_*} f\mu \right| \leq \int_{\mathbb{R}_*} |f|\mu.$$

Moreover, that μ is bounded, possibly signed Radon measure on $\mathbb{R}_*$ and $f \in L^1(\mu)$. Then

$$\left| \int_{\mathbb{R}_*} f\mu \right| \leq \int_{\mathbb{R}_*} |f||\mu|.$$

We will also need Fubini's theorem, also known as the Fubini-Tonelli theorem, which we asssume the reader to be familiar with. The statement and proof of this Fubini's theorem can be found in most modern texts on real analysis. In particular, [7] and [26] contain instructive treatments of the result.

3.5 Weak*-topology and the Banach-Alaoglu Theorem

When we are considering an infinite dimensional Banach space $(X, \|\cdot\|)$, the norm topology on X is too strong for some applications. For example, moving from the finite dimensional setting to the infinite dimensional one makes finding compact sets in the norm topology more difficult. To counter such undesired effects, we may consider other topologies on X, which might be weaker. Such topologies might, for example, allow us to establish the compactness of certain sets more easily. One particularly interesting topology is the weak*-topology. For a more complete treatment of weak*-topology, we refer the reader to [2, 27]

Definition 3.38. If X is a topological vector space, X^* is its dual, $x \in X$ and $x^* \in X^*$, we usually denote the action of x^* on the element x by $\langle x, x^* \rangle \coloneqq x^*(x)$ to suggest the linearity of x^* and a duality between the spaces X and X^*. $\triangleleft$

Consider the space X^* of continuous linear functionals on X. For each element $x \in X$, we have the evaluation map $\beta_x \colon X^* \to \mathbb{K}$ given by $\beta_x(x^*) = \langle x, x^* \rangle$. This map is linear and continuous on X^* equipped with the norm topology. The evaluation maps allow us to define a topology on X^*.

Definition 3.39. The weak*-topology on the space X^* is the topology induced on X^* by the evaluation maps $\beta_x \colon X^* \to \mathbb{K}$. This topology is the weakest topology that makes β_x for every $x \in X$ continuous. The weak*-topology is given by the family of seminorms p_x, where $p_x(x^*) = |\beta_x(x^*)| = |\langle x, x^* \rangle|$.

Remark. It holds that a sequence of functionals $(x_n^*)_{n \in \mathbb{N}}$ in X^* converges to a functional $x^* \in X^*$ in the weak*-topology, in case $\langle x, x_n^* \rangle \to \langle x, x^* \rangle$ for every $x \in X$.

$\triangleleft$

The following proposition provides an equivalent characterization of the weak*-topology on X^*.

Proposition 3.40 (Open sets in weak*-topology)**.** *$U \subset X^*$ is weak*-open, in case for every $x^* \in U$, we can find $\varepsilon > 0$ and elements $x_1, \ldots, x_l \in X$ such that*

$$\bigcap_{i=1}^{l} \{y^* \in X^* \colon |\langle x_i, x^* - y^* \rangle| < \varepsilon\} \subset U.$$

The following theorem, which follows immediately from a proposition in Conway's textbook [2, p. 108], allows us to characterize which functionals from a dual space X^* into the underlying scalar field are continuous.

Proposition 3.41. *Let $\Lambda\colon X^* \to \mathbb{K}$ be a linear functional. Then, it is continuous with respect to the weak*-topology given by the seminorms $p_x, x \in X$ if an only if there are seminorms $p_{x_1}, \dots, p_{x_n}$ and positive scalars $\alpha_1, \dots, \alpha_n$ such that*

$$|\Lambda(x^*)| \leq \sum_{i=1}^{n} \alpha_i p_{x_i}(x^*) = \sum_{i=1}^{n} \alpha_i |\langle x_i, x^* \rangle|.$$

Remark. In this thesis, we only deal with the case $\mathbb{K} = \mathbb{R}$.

Proposition 3.42. *Let $f\colon Y \to X^*$ be a mapping from a metric space (Y, d) to the space X^* (the dual of a normed space X) endowed with the weak*-topology. In case there is a constant C such that for every $x \in X$ we have*

$$p_x(f(y) - f(z)) \leq \|x\| C d(y, z), \qquad \text{for all } y, z \in Y,$$

then f is continuous.

Proof. Let $y \in Y$, and consider the point $f(y) \in X^*$. Let $V \ni f(y)$ be some neighborhood of this point. Therefore, there exists $l \in \mathbb{N}$, elements $x_1, \dots, x_l \in X$ and $\varepsilon > 0$ such that

$$f(y) \in \tilde{V} \coloneqq \bigcap_{i=1}^{l} \{x^*\colon p_{x_i}(f(y) - x^*) < \varepsilon\} \subset V.$$

But now, if we assume $z \in Y$ satisfies $d(y, z) < C(x, \varepsilon) \coloneqq \max(1, \|x_1\|, \dots, \|x_l\|)^{-1} C^{-1} \varepsilon$, then by our assumption $p_{x_i}(f(y) - f(z)) < \varepsilon$ for every $i = 1, \dots, l$. Therefore, if we pick the open ball $U = B(y, C(x, \varepsilon))$, then $fU \subset \tilde{V} \subset V$. Consequently, f is continuous at $y \in Y$, which was arbitrary. □

Given a Banach space X, we define the unit ball of the dual space X^* to be

$$B_{X^*} \coloneqq \{x^* \in X^*\colon \|x^*\| \leq 1\}.$$

The following result, Banach-Alaoglu's theorem, is one of the more important results in functional analysis and its applications. It allows us to find weak*-compact subsets of a dual space very easily.

Theorem 3.43 (Banach-Alaoglu theorem)**.** *Let X be a Banach space. Then the unit ball of the dual space, B_{X^*}, is compact in the weak*-topology.*

Proof. For some nice proofs, see for example [32, p. 29] and [27]. □

Remark. This result is sometimes known simply as Alaoglu's theorem.

The following corollary of the Banach-Alaoglu theorem, which is applicable in cases where X is a separable space, gives us an important tool that will be used later on in this thesis. Notice that this result allows us to pass to sequential compactness when dealing with the unit ball in the dual space with weak*-topology.

Theorem 3.44. *Let X be a separable Banach space. Then the weak*-topology of B_{X^*} is metrizable and this space is weak*-compact.*

Proof. A concise proof is given in [27]. □

3.6 Fréchet derivative of time-dependent functions

In the sequel, we will need a concept of derivative that extends nicely to situations where we consider time-dependent functions with values in some Banach space. One such notion is given by the Fréchet derivative, which we define as follows.

Definition 3.45. For a fixed $T > 0$ and a Banach space X, we say that $f \in C^1([0,T], X)$ in case f is continuous as a map $[0,T] \to X$ and there is a continuous function $\dot{f}\colon [0,T] \to X$, which satisfies the following property: for every $t \in (0,T)$, the Fréchet derivative of f meets $\dot{f}$. Consequently, $\dot{f}$ is given on $(0,T)$ by the function for which

$$\frac{\left\| f(t+\varepsilon) - f(t) - \varepsilon \dot{f}(t) \right\|}{|\varepsilon|} \to 0$$

holds for every $t \in (0,T)$ and whose values in the end points by the left and right limits. ◁

The next result follows immediately from Definition 3.45.

Proposition 3.46. *Suppose that $f\colon [0,T] \to X$ is a function from an interval to a Banach space with the additional property that f possesses the Fréchet derivative $\dot{f}(t)$ on this interval. Then, if we define pointwise for $t \in (0,T)$*

$$f'(t) \coloneqq \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon) - f(t)}{\varepsilon}, \tag{3.4}$$

we have $\dot{f}(t) = f'(t)$ for every $t \in (0,T)$, and hence (by continuous extension), we can identify $\dot{f}(t)$ with $f'(t)$ on the whole interval $[0,T]$.

Next, we introduce notation for spaces of continuous, Banach-valued functions that possess Fréchet-derivative and whose values lie in a prescribed set.

Definition 3.47. Let $T > 0$ be a fixed time length and let $S \subsetneq X$ where X is a Banach space. We say that a function $f\colon [0,T] \to S \subset X$ belongs to the collection $C^1([0,T], S)$ in case $f \in C^1([0,T], X) \cap C([0,T], S)$. In contrast to Definition 3.45, the values of the Fréchet derivative of f need not belong to S. ◁

We now state and prove a result that is analogous to the product rule of differentiable functions.

Proposition 3.48 (Product rule for Fréchet C^1-functions)**.** *Suppose that we have two Banach-valued functions*

$$\begin{aligned} f &\in C^1([0,T], \mathcal{M}_b(\mathbb{R}_*)) \\ \phi &\in C^1([0,T], C_0(\mathbb{R}_*)). \end{aligned}$$

Then, the function

$$\begin{aligned} \Lambda\colon [0,T] &\to \mathcal{M}_b(\mathbb{R}_*) \\ \Lambda(t) &= \phi(\cdot,t) f(\cdot,t) = \phi(t) f(t) \end{aligned}$$

belongs to the collection $C^1([0,T], \mathcal{M}_b(\mathbb{R}_*))$. Moreover, its Fréchet derivative is the map

$$\begin{aligned} &[0,T] \to \mathcal{M}_b(\mathbb{R}_*) \\ &\dot{\Lambda}(t) = \dot{\phi}(\cdot,t) f(\cdot,t) + \phi(\cdot,t) \dot{f}(\cdot,t). \end{aligned}$$

Proof. Consider the Radon measure -valued mapping.

$$\begin{aligned} t &\mapsto \phi(t)f(t) \\ \mathbb{R}_+ &\to \mathcal{M}_b(\mathbb{R}_*) \end{aligned}$$

Here $\phi(t)f(t)$ is a Radon measure, since any Radon measure multiplied by a continuous function is a Radon measure.

For a fixed $t \in (0,T)$, and any sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned} &f(t+\varepsilon)\phi(t+\varepsilon) - f(t)\phi(t) - \varepsilon\dot{f}(t)\phi(t) - \varepsilon f(t)\dot{\phi}(t) \\ &= f(t+\varepsilon)\left(\phi(t+\varepsilon) - \phi(t) - \varepsilon\dot{\phi}(t)\right) \\ &+ \phi(t)\left(f(t+\varepsilon) - f(t) - \varepsilon\dot{f}(t)\right) \\ &+ \varepsilon f(t+\varepsilon)\dot{\phi}(t) - \varepsilon f(t)\dot{\phi}(t) \end{aligned}$$

Here $\left\|f(t+\varepsilon)\dot{\phi}(t) - f(t)\dot{\phi}(t)\right\| \to 0$ as $\varepsilon \to 0$ in the norm topology of $\mathcal{M}(\mathbb{R}_*)$, since f is continuous on the interval $(0,T)$ and $\dot{\phi}(t) \in C_0(\mathbb{R}_*)$ is a bounded function. On the other hand,

$$|\varepsilon|^{-1}\|f(t+\varepsilon)\|\left\|\phi(t+\varepsilon) - \phi(t) - \varepsilon\dot{\phi(t)}\right\| \to 0,$$

since $\sup_{t\in[0,T]} \|f\| < \infty$. Similarly

$$|\varepsilon|^{-1}\|\phi(t)\|\left\|f(t+\varepsilon) - f(t) - \varepsilon\dot{f}(t)\right\| \to 0.$$

Combining the above three limits and using triangle inequality on the norm, we get that for each $t \in (0,T)$

$$\frac{\left\|f(t+\varepsilon)\phi(t+\varepsilon) - f(t)\phi(t) - \varepsilon(\dot{f}(t)\phi(t) + f(t)\dot{\phi}(t))\right\|}{|\varepsilon|} \to 0 \tag{3.5}$$

as $\varepsilon \to 0$. This means precisely that the Fréchet derivative of the map $t \mapsto \phi(t)f(t)$ is the function

$$\begin{aligned} t &\mapsto \phi(t)\dot{f}(t) + \dot{\phi}(t)f(t) \\ \mathbb{R}_+ &\to \mathcal{M}_b(\mathbb{R}_*) \end{aligned} \tag{3.6}$$

After this, we can extend the derivative function from $(0,T)$ to the closed interval $[0,T]$ by taking left and right limits. These exists and are finite by our assumption that $f \in C^1([0,T],\mathcal{M}_b(\mathbb{R}_*))$ and $\phi \in C^1([0,T],C_c(\mathbb{R}_+))$. Consequently, $\phi f \in C^1([0,T],\mathcal{M}_b(\mathbb{R}_*))$, and its derivative is $\dot{\phi} f + \phi \dot{f}$, which was to be shown. □

We will also need the following result, which allows us to differentiate integrals of time-dependent C^1-function over C^1 time-dependent measures

Proposition 3.49 (Differentiating integral)**.** *Assume that functions $\phi \in C^1([0,T],C_0(\mathbb{R}_*))$, and $f \in C^1([0,T],\mathcal{M}_b(\mathbb{R}_*))$ are given. Consider the mapping*

$$\begin{aligned} t &\mapsto \int_{\mathbb{R}_*} \phi(x,t) f(\mathrm{d}x,t) \\ [0,T] &\to \mathbb{R} \end{aligned}$$

Then this mapping is differentiable on the open interval $(0,T)$, and its derivative is given by

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_*} \phi(x,t) f(\mathrm{d}x,t) \\ &= \int_{\mathbb{R}_*} \phi(x,t) \dot{f}(\mathrm{d}x,t) + \int_{\mathbb{R}_*} \dot{\phi}(x,t) f(\mathrm{d}x,t). \end{aligned} \tag{3.7}$$

Moreover, since the right hand side is continuous and finite on $[0,T]$, the derivative can be continuously extended to the whole closed interval $[0,T]$. Therefore, $\int_{\mathbb{R}_*} \phi(x,t) f(\mathrm{d}x,t) \in C^1([0,T],\mathbb{R})$

Proof. We note that if we are given such functions ϕ, f with

$$\begin{aligned} \phi &\in C^1([0,T],C_0(\mathbb{R}_*)) \\ f &\in C^1([0,T],\mathcal{M}_b(\mathbb{R}_*)), \end{aligned}$$

then the pointwise defined product ϕf of these functions is a map that associates each $t \in (0,T)$ to a finite signed measure. In other words, for each $t \in (0,T)$, the value of ϕf, $(\phi f)(t) = \phi(t) f(t)$, is a finite Radon measure.

By Proposition 3.48, we have that this product function ϕf satisfies $\phi f \in C^1([0,T],\mathcal{M}_b(\mathbb{R}_*))$. Let us denote the function ϕf for time by g. Since g is a Radon measure -valued function,

for each fixed $t \in (0,T)$ we can integrate with respect to $g(t)$. We now want to prove the following non-trivial identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_*} g(\mathrm{d}x,t) = \int_{\mathbb{R}_*} \dot{g}(\mathrm{d}x,t) \tag{3.8}$$

To do this, we want to show that the time derivative of $\int_{\mathbb{R}_*} g(\mathrm{d}x,t)$ matches the integral over the Fréchet derivative $\int_{\mathbb{R}_*} \dot{g}(\mathrm{d}x,t)$ for $t \in (0,T)$. After this, we can extend this function to the closed interval $[0,T]$ by taking right and left limits.

Indeed, we have

$$\begin{aligned}
&\frac{\left|\int_{\mathbb{R}_*} g(\mathrm{d}x,t+\varepsilon) - \int_{\mathbb{R}_*} g(\mathrm{d}x,t) - \varepsilon\int_{\mathbb{R}_*}\dot{g}(\mathrm{d}x,t)\right|}{|\varepsilon|}\\
&= \frac{|g(t+\varepsilon)(\mathbb{R}_*) - g(t)(\mathbb{R}_*) - \varepsilon\dot{g}(t)(\mathbb{R}_*)|}{|\varepsilon|}\\
&= \frac{|(g(t+\varepsilon) - g(t) - \varepsilon\dot{g}(t))(\mathbb{R}_*)|}{|\varepsilon|}\\
&\leq \frac{\|g(t+\varepsilon) - g(t) - \varepsilon\dot{g}(t)\|}{|\varepsilon|}\\
&\xrightarrow{\varepsilon\to 0} 0.
\end{aligned}$$

Since here we assumed that $\dot{g}$ is the Fréchet derivative of g.

Consequently, we have the equality $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_*} g(\mathrm{d}x,t) = \int_{\mathbb{R}_*}\dot{g}(\mathrm{d}x,t)$, which can be extended continuously to cover the end points 0 and T. Since $g = \phi f$ and $\dot{g} = \dot{\phi}f + \phi\dot{f}$, we have the desired identity. □

With these mathematical concepts and results at our disposal, we now move on to consider the formulation of coagulation equations with injection.

4 Reformulating the problem

In the present chapter, we shall first lay down a suitable setting within which we are able to prove the existence theorem related to the regularized coagulation equations with injection. After this, we make a reformulation of the statement, which will eventually lead to the desired result. Since we are interested in more general injection terms and initial data than continuous functions, we need to define suitable spaces in which our initial data, injections and solutions live in.

Much of this and the next chapter follows the article [6]. Indeed, the present author was involved in supplementing some of the technical details to the proofs mentioned in this article.

To recapitulate, we are trying to establish the existence of stationary, non-equilibrium solutions to a regularized coagulation equation. At this point, we have not defined what we mean by the regularized coagulation equation, but this will be defined in this section. Moreover, we include an injection term, which pushes new particles to the system, and contributes non-negatively to the time-evolution of the measure. We are trying to find solutions where the input from the injection term flows up in the particle size distribution at the same rate, whereby the measure remains constant.

As a first approximation, suppose that we are trying to solve the problem for a non-regularized coagulation equation with injection. Assuming that the particle size distribution is given by a function, this would mean trying to find a stationary solution to the following evolution equation.

$$\partial_t f(x,t) = \mathcal{C}f(x,t) + \eta(x), \qquad \text{for all } x \in \mathbb{R}_+ \tag{4.1}$$

where the coagulation operator $\mathcal{C}$ is given by

$$\mathcal{C}f(x,t) = \frac{1}{2}\int_0^x K(x-y,y)f(x-y,t)f(y,t)\,\mathrm{d}y - \int_0^\infty K(x,y)f(x,t)f(y,t)\,\mathrm{d}y.$$

However, this equation is not quite suitable for our purposes. We would like to allow the injection term η to be concentrated on a single point $a \in \mathbb{R}_*$. This already forces us to move away from considering spaces of continuous function. Additionally, to be able to fully utilise various tools from functional analysis, it makes sense to allow the injection term η to be a compactly supported, positive Radon measure. If we do this, the pointwise definition seen in (4.1) doesn't make sense anymore.

Furthermore, instead of directly trying to find a solution to something like (4.1), we will first regularize the coagulation operator, which will enable us to use powerful tools from functional analysis.

4.1 Making sense of the function spaces

As mentioned in the beginning of this section, we will move away from considering the coagulation operator $\mathcal{C}$ as operating on the space $C(\mathbb{R}_+ \times \mathbb{R}_*)$. Instead, we assume that the time-dependent size distribution objects f are functions that assign every point in time $t \in \mathbb{R}_+$ to a Radon measure $f(t)$. At each point in time, this Radon measure tells us the amount of particles with size inside measurable sets. Additionally, we assume that all particles are of strictly positive size, so the Radon measures should be defined on $\mathbb{R}_* = (0, +\infty)$. As a natural space of particle size distributions, we are then then led to consider the space of positive and bounded Radon measures $\mathcal{M}_{+,b}(\mathbb{R}_*) = \mathcal{M}_+(\mathbb{R}_*) \cap \mathcal{M}_b(\mathbb{R}_*)$. Forthunately for us, this is a closed subset of the Banach space $\mathcal{M}_b(\mathbb{R}_*)$.

Recall that the space of signed (bounded) Radon measures satisfies the isometric isomorphism $\mathcal{M}_b(\mathbb{R}_*) \cong (C_0(\mathbb{R}_*))^*$, and that each positive functional corresponds to a unique positive Radon measure. This allows us to treat the Radon measures on $\mathbb{R}_*$ as positive continuous linear functionals on the space $C_0(\mathbb{R}_*)$.

Formulating the problem in terms of Radon measures raises new problems, though. We cannot simply try to find solutions to the problem

$$\partial_t f(x,t) = \mathcal{C}[f](x,t) + \eta(x),$$

since the measures $f(t)$ and η don't necessarily allow for a density function and hence might not be definable pointwise. The coagulation equation must be tweaked a bit in order for it to make sense in the more general setting of Radon measures.

What we can do is integrate both sides of (4.1) against a test function $\phi \in C_c(\mathbb{R}_*)$, and formally (evoking Fubini's theorem without justification, since we don't know if the integral diverges) rearrange the first term on the right hand side with a change of variables. This brings us to a weak formulation [see 16] of the coagulation equation with injection,

namely

$$\begin{aligned}&\int_{\mathbb{R}_*} \phi(x)\partial_t f(\mathrm{d}x,t)\\ &= \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\left(\phi(x+y)-\phi(x)-\phi(y)\right) f(\mathrm{d}x,t)f(\mathrm{d}y,t) + \int_{\mathbb{R}_*}\phi(x)\eta(\mathrm{d}x).\end{aligned} \tag{4.2}$$

Moving to this formulation is justified by Fubini's theorem, in case the double integral is finite. We will refer to this equation as the weak coagulation equation, but for our purposes, we still have to modify the equation a bit by introducing a regularization function as a factor in front of $\phi(x+y)$. The regularization term prevents the mass of the evolving measure from escaping a compact interval [6]. The regularization function will depend on the given injection measure η, so we will discuss the regularization more closely in the next subsection, where the class of valid injection measures is defined.

4.2 Time-dependent and stationary solutions

In the previous subsection, we clarified which solution space we are considering in our reformulation of the regularized coagulation equation. We still need to formulate what we mean by solutions to the regularized coagulation equation with injection. Now, we try to formulate some conditions on the injection measure, coagulation kernel and initial data. This allows us to give a definition of time-dependent solutions of the coagulation equation with injection.

We want to specify which injection measures are admissible. It is physically motivated that there is some bound on the size of the particles that we can input into the system. Moreover, we assume that there is some lower bound on the inputted particles, and we let this size to be 1. Mathematically, this amounts to requiring that η satisfies $\mathrm{supp}(\eta) \subset [1, C_\eta]$. This means that the size of the particles injected into the system is bounded from above and below.

Definition 4.1 (Admissible injection measures)**.** We call an injection measure $\eta \in \mathcal{M}_{+,b}(\mathbb{R}_*)$ *admissible*, if $\mathrm{supp}(\eta) \subset [1, C_\eta]$ for some $C_\eta \geq 1$ [6]. That is, admissible measures are finite, positive Radon measures that are supported on the compact interval $[1, C_\eta]$.

Remark. The assumption $\eta \in \mathcal{M}_{+,b}(\mathbb{R}_*)$ guarantees that the norm of η satisfies

$$\|\eta\| = |\eta|(\mathbb{R}_*) = \eta(\mathbb{R}_*) < \infty.$$

◁

Recall that our coagulation operator involves a kernel function, which intuitively tells us how fast particles of certain sizes coalesce to form bigger particles. We now define those coagulation kernels, for which we can prove the theorem. The collection of these kernels depends on the fixed injection measure η.

Definition 4.2 (Good coagulation kernels)**.** We say that a coagulation kernel $K\colon \mathbb{R}_* \times \mathbb{R}_* \to [0, +\infty)$ belong to a class G_C of good coagulation kernels, in case $K \in C(\mathbb{R}_* \times \mathbb{R}_*)$, K is symmetric ($K(x, y) = K(y, x)$), there exists some compact set $D \subset \mathbb{R}_* \times \mathbb{R}_*$ such that $\operatorname{supp}(K) \subset D$, and if for some $L > C_\eta > 0$, $K(x, y) \in [a_1, a_2]$, for $(x, y) \in [1, 2L]^2$, $D \subset [1, 4L]^2$, where $0 < a_1 < a_2 < \infty$, and finally if $\|K\|_\infty \leq a_2$. ◁

To get to the regularized coagulation equation, our next step is to define a cut-off function that regularizes the coagulation operator. For a fixed $L > 0$, which is the same one appearing in Definition 4.2, we let $\zeta_L\colon \mathbb{R}_* \to [0, 1]$ to be a continuous, compactly supported function, with $\zeta_L = 1$ on $(0, L]$ and $\zeta_L = 0$ on $\mathbb{R}_* \setminus (0, 2L)$.

With these definitions in our disposal, we can now formulate the concept of time-dependent solution of a regularized coagulation equation with injection.

Definition 4.3 (Time-dependent solution)**.** Let η be an admissible injection measure. Let $L > C_\eta$, and suppose that K is a good coagulation kernel, and $f_0 \in \mathcal{M}_{+,b}(\mathbb{R}_*)$ is some positive, finite initial measure, which satisfies $f_0((0, 1) \cup (2L, +\infty)) = 0$. We say that a measure-valued function $f \in C^1([0, T], \mathcal{M}_{+,b}(\mathbb{R}_*))$ is a *time-dependent* solution to the regularized coagulation equation on the interval $[0, T]$ with the initial measure f_0 in case the following three conditions are verified:

1. The function f matches the given initial data in the sense that $f(0) = f_0$.

2. The function f is uniformly bounded,

$$\sup_{t\in[0,T]} \left(\int_{\mathbb{R}_*} f(\mathrm{d}x, t) \right) < \infty.$$

3. For any continuously differentiable test function $\phi \in C^1([0, T], C_c(\mathbb{R}_*))$ the following

identity holds

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_*}\phi(x,t)f(\mathrm{d}x,t)-\int_{\mathbb{R}_*}\dot{\phi}(x,t)f(\mathrm{d}x,t)\\
&=\frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}K(x,y)\left[\phi(x+y,t)\zeta_L(x+y)-\phi(x,t)-\phi(y,t)\right]f(\mathrm{d}x,t)f(\mathrm{d}y,t)\\
&+\int_{\mathbb{R}_*}\phi(x,t)\eta(\mathrm{d}x)\\
&=\langle\phi(t),\mathcal{C}_L[f](t)\rangle+\langle\phi(t),\eta\rangle\,. \qquad (4.3)
\end{aligned}$$

4. $f((0,1)\cup(2L,+\infty),t)=0$ for every $t\in[0,T]$.

Remark. We note that, sending $x=x'-y$, and using Fubini's theorem, we get

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}K(x,y)\phi(x+y,t)\zeta_L(x+y)f(\mathrm{d}x,t)f(\mathrm{d}y,t)\\
&=\frac{1}{2}\int_{\mathbb{R}_*}\int_{(y,\infty)}K(x'-y,y)\zeta_L(x')\phi(x',t)f(\mathrm{d}x',t)f(\mathrm{d}y,t)\\
&=\frac{1}{2}\int_{\mathbb{R}_*}\int_{(0,x')}K(x'-y,y)\zeta_L(x')\phi(x',t)f(\mathrm{d}y,t)f(\mathrm{d}x',t).
\end{aligned}$$

By a change of notation, this can be written as

$$\frac{1}{2}\int_{\mathbb{R}_*}\int_{(0,x)}K(x-y,y)\zeta_L(x)\phi(x)f(\mathrm{d}y)f(\mathrm{d}x).$$

This motivates the definition of a weak solution to the time-dependent regularized coagulation equation, given in 4.3.

◁

To understand the above definition, recall that the results in the mathematical preliminaries section imply the following corollary.

Corollary 4.4. *Suppose that $f\in C^1([0,T],\mathcal{M}_{+,b}(\mathbb{R}_*))$ and $\phi\in C^1([0,T],C_0(\mathbb{R}_*))$. Then*

$$\int_{\mathbb{R}_*}\phi(x,t)\dot{f}(\mathrm{d}x,t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_*}\phi(x,t)f(\mathrm{d}x,t)-\int_{\mathbb{R}_*}\dot{\phi}(x,t)f(\mathrm{d}x,t). \qquad (4.4)$$

Proof. We apply Proposition 3.49 to the map given by $t \mapsto \phi(t)f(t)$, which belongs to the collection $C^1([0,T], \mathcal{M}_{+,b}(\mathbb{R}_*))$ by Proposition 3.48. After applying this proposition once more, we arrive at the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_*} \phi(x,t)f(\mathrm{d}x,t) = \int_{\mathbb{R}_*} \dot{\phi}(x,t)f(\mathrm{d}x,t) + \int_{\mathbb{R}_*} \phi(x,t)\dot{f}(\mathrm{d}x,t)$$

All integrals in the above expression are finite by our assumptions on ϕ and f. Therefore, we can rearrange the terms in the above equality to get (4.4). □

Next, we define what it means for a Radon measure to be a stationary solution to the regularized coagulation equation.

Definition 4.5. Fix two positive, finite Radon measures $f, \eta \in \mathcal{M}_{+,b}(\mathbb{R}_*)$. For every test function $\phi \in C_c(\mathbb{R}_*)$, we define assignments

$$\langle \phi, \mathcal{C}[f]_L \rangle = \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\left(\phi(x+y)\zeta_L(x+y) - \phi(x) - \phi(y)\right) f(\mathrm{d}x)f(\mathrm{d}y),$$

and

$$\langle \phi, \mathcal{I}[\eta] \rangle = \int_{\mathbb{R}_*} \phi(x)\eta(\mathrm{d}x).$$

◁

Definition 4.6 (Stationary solution)**.** Given an admissible measure η and a good kernel K, we call a measure $f \in \mathcal{M}_{+,b}(\mathbb{R}_*)$ a *stationary solution* of the regularized coagulation equation, in case we have

$$\langle \phi, \mathcal{C}[f]_L \rangle + \langle \phi, \mathcal{I}[\eta] \rangle = 0 \tag{4.5}$$

for every test function $\phi \in C_c(\mathbb{R}_*)$.

◁

Definition 4.6 captures what we mean when we say that a measure valued function is a stationary solution to the regularized coagulation equation. In the next section, we show that if η is an admissible injection kernel and if K is in the family of good coagulation kernels G_C, then there exists a stationary solution to the coagulation equation with injection. More precisely:

Theorem 4.7. *Suppose that η is an admissible injection measure in the sense of Definition 4.1. Suppose further that $K \in G_C$ in the sense of Definition 4.2. Then there exists a stationary solution $f \in \mathcal{M}_{+,b}(\mathbb{R}_*)$ to the regularized coagulation equation with injection in the sense of Definition 4.6.*

4.3 Outline of the proof

We will give the proof of Theorem 4.7 in the next section. Since the proof involves multiple different steps, for the sake of clarity we briefly outline the main tasks, which follow those laid out in [6]. Indeed, this thesis consists mainly of writing out in greater detail all the steps in the mentioned article.

On a conceptual level, we can say that the proof consists of two main parts, both of which break down to a number of smaller parts. In the first part, we will show that every initial value problem with suitable injection measure η, kernel K and initial data f_0 has a global time-dependent solution. To this end, we fix a measure η and a kernel K. The first steps of this part consist of building a suitable contractive operator $\mathcal{F}$ on the complete, metrisable subset $C([0,T],\mathscr{X}_L)$ of the Banach space $C([0,T],\mathcal{M}_b(\mathbb{R}_*))$, where

$$\mathscr{X}_L \coloneqq \{\mu \in \mathcal{M}_{+,b}(\mathbb{R}_*) : \mu((0,1)\cup(2,L)) = 0\}.$$

For a suitably small time interval $[0,T]$, we can find the unique fixed point of this mapping using Banach's fixed point theorem. This operator captures the coagulation equation in the sense that its fixed points are unique time-dependent solutions to the regularized coagulation equation. In the above steps, we use the Riesz-Markov-Kakutani theorem to show that $\mathcal{F}$ is a mapping into the correct space.

If we denote by $f\colon [0,T] \to \mathscr{X}_L$ the solution to the time-dependent equation on the interval $[0,T]$, then we can use estimates on the evolution of the norm $\|f(t)\|$ to extend this solution from $[0,T]$ to $\mathbb{R}_+$. Therefore, we have shown that for each initial data f_0, there exists a function that solves the time-dependent (regularized) coagulation equation for all times. This concludes the first main part of the proof.

In the second part of the proof, we use the existence of time-dependent solutions to show that there exist stationary solutions. Indeed, since for every initial value f_0 there exists a time-dependent solution f to the regularized coagulation equation with injection, we can define a semigroup of operators $\{S(t)\}_{t\geq 0}$, where $S(t)$ maps a given initial value f_0 to the measure $f(t)$. This mapping is continuous in the weak*-topology and the mapping $t \mapsto S(t)f_0$ is continuous, when the codomain is equipped with the weak*-topology. Moreover, there is a non-empty, convex, compact set that is invariant under $\{S(t)\}_{t\geq 0}$. Pointwise (in t) application of Schauder's fixed point theorem together with the continuity results and the semigroup property allows us to show that there is an initial value f_0 satisfying $S(t)f_0 = f_0$. This measure can be shown to satisfy the definition of a stationary solution.

5 Proof of the Existence Result for Regularized 1D Coagulation Equations with Injection

In the present chapter, we move on to prove the main mathematical result in the thesis, namely the existence of stationary, non-trivial solutions to the regularized coagulation equation with injection, provided that the injection term meets certain assumptions and the coagulation kernel is compactly supported on $\mathbb{R}_*^2$ and sufficiently nice. The treatment of the proof follows the recent article by Ferreira et al. [6], where the result was first establihed.

As was stated in the previous section, we assume that we are given an injection term $\eta \in \mathcal{M}_{+,b}(\mathbb{R}_*)$, which is supported on some compact interval $[1, C_\eta]$. Moreover, we assume that we are given a good coagulation kernel

$$K \colon \mathbb{R}_* \times \mathbb{R}_* \longrightarrow \mathbb{R}_+,$$

which is nonnegative, symmetric, and continuous. To be more precise, we assume that we are given a cut-off constant $L > C_\eta > 0$ and bounding constants $0 < a_1 \leq a_2$, for which the following estimates and identities hold: $K(x, y) \in [a_1, a_2]$ for $(x, y) \in [1, 2L]^2$, and $K(x, y) = 0$ for $x \geq 4L$ or $y \geq 4L$. For the intermediate values $x, y \in (2L, 4L)$, we assume without losing generality that $K(x, y) \leq a_2$.[7] All in all, these assumption amount to requiring that K is a compactly supported, continuous, nonnegative function on $\mathbb{R}_* \times \mathbb{R}_*$, which has explicit upper and lower bounds on the set $[1, 2L]^2$.

In what follows, we will prove the existence of stationary solutions according to Definition 4.6. There are two main parts of the proof, and we will split the proof into two subsections accordingly. The first subsection one deals with establishing the existence of time-dependent solutions in sense of Definition 4.3 for arbitrarily long times. In the second subsection, we will use the existence claim to define a semigroup of operators on a carefully picked set of initial values. This semigroup evolves the initial values following the unique time-dependent solution that we have found. We can use the properties of this semigroup to find a stationary solution.

[7] This can be done, since $K \in C_0(\mathbb{R}_*^2)$, so the sup-norm of the kernel is bounded by some constant $\|K\| \coloneqq \sup_{(x,y)\in[1,4L]^2} |K(x, y)|$. Therefore, we can just select $a_2 \coloneqq \|K\|$ in the first place.

5.1 Time-dependent solutions to initial value problems

Recall that we have fixed an andmissible injection measure η with its support contained in $[1, C_\eta]$. Fix a constant $L > C_\eta$, let K be a good kernel and f_0 a valid initial measure with L as the cut-off constant.

The techniques we are about to use allow us to find solutions on certain compact time intervals $[0, T]$. As a first step, we want to find a measure-valued function $f \in C^1([0,T], \mathcal{M}_{+,b}(\mathbb{R}_*))$ that satisfies this equation on the finite interval $[0, T]$, with some additional estimates on the size of the derivative of the function.

Corresponding to any map $f \in C([0,T], \mathcal{M}_{+,b}(\mathbb{R}_*))$, we associate a function

$$\begin{aligned} a[f] \colon \mathbb{R}_* \times [0,T] &\to \mathbb{R}_+ \\ a[f](x,s) &= \int_{\mathbb{R}_*} K(x,y) f(\mathrm{d}y, s). \end{aligned}$$

For any such f, the function $a[f]$ is continuous in x and s. It is also non-negative, since $f(s)$ is a positive Radon measure for every $s \in [0,T]$ and K is a non-negative function.

Recall that for $f \in C([0,T], \mathcal{M}_{+,b}(\mathbb{R}_*))$, we have the uniform bound

$$\|f\|_T \coloneqq \sup_{t \in [0,T]} \|f(t)\| < \infty, \tag{5.1}$$

since $[0,T]$ is a compact interval. Per our assumptions on the kernel, we also have the uniform bound $K(x,y) \leq a_2$. It follows that the value $a[f](x,s)$ is bounded from above by $a_2 \int_{\mathbb{R}_*} f(\mathrm{d}x, s)$. Therefore, by (5.1), we have

$$\left| \int_0^t a[f](x,s) \, \mathrm{d}s \right| \leq t a_2 \|f\|_T,$$

so the expression on the left hand side can be made small uniformly in $x \in \mathbb{R}_*$ by requiring $t > 0$ to be small enough.

Let us define a mapping $\mathcal{F}$ that takes a measure valued function $f \in C([0,T], \mathcal{M}_{+,b}(\mathbb{R}_*))$, and returns a time-dependent real-valued functional acting on the space $C_0(\mathbb{R}_*)$ by the following rule:

$$\langle \phi, \mathcal{F}[f](t) \rangle \coloneqq \langle \phi, \mathcal{F}_1[f](t) \rangle + \langle \phi, \mathcal{F}_2[f](t) \rangle + \langle \phi, \mathcal{F}_3[f](t) \rangle \,, \tag{5.2}$$

where the three maps on the right hand side of the equation are defined on $C_0(\mathbb{R}_*)$ for each $t \in [0,T]$ by the following three assignments:

$$\begin{aligned}
\langle \phi, \mathcal{F}_1[f](t) \rangle &\coloneqq \int_{\mathbb{R}_*} \phi(x) e^{-\int_0^t a[f](x,s)\,\mathrm{d}s} f_0(\mathrm{d}x) \\
\langle \phi, \mathcal{F}_2[f](t) \rangle &\coloneqq \frac{1}{2} \int_0^t \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} K(x,y)\phi(x+y) h_L(x+y,s) f(\mathrm{d}x,s) f(\mathrm{d}y,s)\,\mathrm{d}s \\
\langle \phi, \mathcal{F}_3[f](t) \rangle &\coloneqq \int_{\mathbb{R}_*} \phi(x) \left(\int_0^t e^{-\int_s^t a[f](x,\xi)\,\mathrm{d}\xi}\,\mathrm{d}s \right) \eta(\mathrm{d}x).
\end{aligned}$$

Here $h_L(x,s) = \zeta_L(x) e^{-\int_s^t a[f](x,\xi)\,\mathrm{d}\xi}$.

The mapping $\mathcal{F}$, which looks quite complicated, helps us find a stationary solution to the regularized coagulation with injection. Our aim is to find a point that is fixed under $\mathcal{F}$. The first step towards this is making sure that for each measure-valued function $f \in C([0,T], \mathcal{M}_{+,b}(\mathbb{R}_*))$ and for each $t \in [0,T]$, the functional $\mathcal{F}[f](t)$, which acts on the space of test functions $C_0(\mathbb{R}_*)$, defines a measure in $\mathcal{M}_{+,b}(\mathbb{R}_*)$. This can be achieved by an application of the complex Riesz-Markov-Kakutani representation theorem 3.32. To apply this theorem, we have to ascertain that $\mathcal{F}[f](t)$ is a positive, continuous linear functional on the space $C_c(\mathbb{R}_*)$, which then allows us to extend it uniquely to the space $C_0(\mathbb{R}_*)$.

Lemma 5.1. *The mapping $\mathcal{F}[f](t)$ is a continuous linear functional from $C_c(\mathbb{R}_*)$ to $\mathbb{R}$. Moreover, it is a positive functional in the sense that it maps every positive element $\phi \geq 0$ of $C_c(\mathbb{R}_*)$ to a nonnegative real number.*

Proof. First, we want to check that we have $\mathcal{F}_i[f](t) \colon C_c(\mathbb{R}_*) \to \mathbb{R}$ for $i = 1,2,3$. Recall that

$$\langle \phi, \mathcal{F}_1[f](t) \rangle = \int_{\mathbb{R}_*} \phi(x) e^{-\int_0^t a[f](x,s)\,\mathrm{d}s} f_0(\mathrm{d}x).$$

This integral is well-defined, since the function $x \mapsto g(x;t) \coloneqq e^{-\int_0^t a[f](x,s)\,\mathrm{d}s}$ is continuous in x, and consequently Borel-measurable. Hence, for every t, the function $x \mapsto \phi(x) g(x;t)$ is Borel-measurable, and its integral with respect to the Radon measure $f_0 = f(\cdot, 0)$ is well-defined. We note that $\int_0^t a[f](x,s)\,\mathrm{d}s \geq 0$, so the exponential factor is bounded from above by 1. Therefore, the finiteness of the expression follows from the boundedness of ϕ and $\int_{\mathbb{R}_*} f(\mathrm{d}x, 0)$.

Similarly, in the expression

$$\langle \phi, \mathcal{F}_3[f](t) \rangle = \int_{\mathbb{R}_*} \phi(x) \left(\int_0^t e^{-\int_s^t a[f](x,\xi)\,\mathrm{d}\xi}\,\mathrm{d}s \right) \eta(\mathrm{d}x),$$

the function inside the integral is both Borel-measurable and finite. To show that $\mathcal{F}[f](t)$ is well defined, we still need to show that $\mathcal{F}_2[f](t)$ is well defined. Recall that

$$\langle \phi, \mathcal{F}_2[f](t) \rangle = \frac{1}{2} \int_0^t \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} K(x,y)\phi(x+y)h_L(x+y,s) f(\mathrm{d}x,s) f(\mathrm{d}y,s)\,\mathrm{d}s$$

First of all for fixed y and s,

$$x \mapsto h_L(x+y,s) = \frac{\zeta_L(x+y)}{2} \phi(x+y) K(x,y) e^{-\int_s^t a[f](x+y,\xi)\,\mathrm{d}\xi}$$

is continuous and hence Borel-measurable. Thus, the innermost integral is defined. It is also finite, since the integrand is bounded and $f(\cdot, s)$ is a finite measure.

For a fixed $s \in [0,T]$, the map

$$y \mapsto \int_{\mathbb{R}_*} \frac{\zeta_L(x+y)}{2} K(x,y)\phi(x+y) e^{-\int_s^t a[f](x+y,\xi)\,\mathrm{d}\xi} f(\mathrm{d}x,s)$$

is continuous, and thus Borel-measurable. This means that the second innermost integral in $\langle \phi, \mathcal{F}_2[f](t) \rangle$ is well defined. Finally, the map

$$s \mapsto \int_{\mathbb{R}_*} f(\mathrm{d}y,s) \int_{\mathbb{R}_*} \frac{\zeta_L(x+y)}{2} K(x,y)\phi(x+y) e^{-\int_s^t a[f](x,\xi)\,\mathrm{d}\xi} f(\mathrm{d}y,s)$$

is continuous by the continuity of f, and so it is Borel-measurable. Also, the integral is finite, since $\|f\|_T < \infty$, $\|\phi\| < \infty$ and we are integrating over a finite time interval. Thus, $\mathcal{F}_2[f](t) \colon C_c(\mathbb{R}_*) \to \mathbb{R}$ is defined.

Combining the above facts gives us the first desired result: $\mathcal{F}[f](t)$ is a well defined map $C_c(\mathbb{R}_*) \to \mathbb{R}$. We still need to show that it is linear, positive and continuous.

Clearly each functional $\mathcal{F}_i[f](t)$ is linear, so the functional $\mathcal{F}[f](t)$, being their sum, is linear as well. Continuity of the functional follows once we show that each functional $\mathcal{F}_i[f] \colon C_c(\mathbb{R}_*) \to \mathbb{R}$ is continuous. To this end, recall that f_0 is a positive measure, so the

total variation measure of f_0 is the same as f_0, in other words $|f_0| = f_0$. Now,

$$\begin{aligned} |\langle \phi - \psi, \mathcal{F}_1[f](t)\rangle| &= \left| \int_{\mathbb{R}_*} (\phi(x) - \psi(x)) f_0(\mathrm{d}x) e^{-\int_0^t a[f](x,s)\,\mathrm{d}s} \right| \\ &\leq \|\phi - \psi\| \int_{\mathbb{R}_*} f_0(\mathrm{d}x) \\ &= \|\phi - \psi\| \|f_0\|. \end{aligned} \tag{5.3}$$

The product on the right hand side can be made smaller than any prescribed $\varepsilon > 0$ by taking $\|\phi - \psi\| < \frac{\varepsilon}{1+\|f_0\|}$. This means that $\mathcal{F}_1[f](t)$ is continuous.

By a similar computation, we estimate

$$|\langle \phi, \mathcal{F}_3[f](t)\rangle - \langle \psi, \mathcal{F}_3[f](t)\rangle| \leq t\|\phi - \psi\|\|\eta\|. \tag{5.4}$$

This establishes the continuity of $\mathcal{F}_3[f](t)$ for each fixed $t \in \mathbb{R}_+$.

Lastly, we want to show that the functional $\mathcal{F}_2[f](t)$ is continuous. Recall that for a fixed $T \geq 0$, a simple computation yields that

$$|\langle \phi - \psi, \mathcal{F}_2[f](t)\rangle| \leq T a_2 \|\phi - \psi\| \|f\|_T^2. \tag{5.5}$$

Since a_2 and T are constant in this context, the right hand side can be made smaller than any given $\varepsilon > 0$ by requiring that $\|\phi - \psi\|$ is sufficiently small. Thus, $\mathcal{F}_2[f](t)$ is continuous. The above three continuity results together imply that the functional $\mathcal{F}[f](t)$ is continuous as a map $C_c(\mathbb{R}_*) \to \mathbb{R}$

The positivity of the resulting functional follows simply from the fact that if $\phi \in C_c(\mathbb{R}_*) \geq 0$, then all the functions and measures appearing in the integrals $\mathcal{F}_i[f](t)$ are nonnegative and finite. Consequently, $\langle \phi, \mathcal{F}[f](t)\rangle \in \mathbb{R}$ is a nonnegative real number. □

Remark. Up to this point, we have shown that $\mathcal{F}[f](t)\colon C_c(\mathbb{R}_*) \to \mathbb{R}$ is a continuous, positive, linear functional on the space $C_c(\mathbb{R}_*)$. By the basic Banach space theory, this functional has a unique extension to a continuous, positive, linear functional on the Banach space $C_0(\mathbb{R}_*)$, which is the completion of $C_c(\mathbb{R}_*)$ in the sup-norm. We will then simply denote this extension by the same symbol, $\mathcal{F}[f](t)$.

We introduce a shorthand for the set of positive, bounded Radon measures, which vanish outside the compact interval $[1, 2L]$:

$$\mathscr{X}_L \coloneqq \{\mu \in \mathcal{M}_{+,b}(\mathbb{R}_*)\colon \mu((0,1) \cup (2L, +\infty)) = 0\}.$$

By definition, this is a subset of the set of positive, bounded Radon measures $\mathcal{M}_{+,b}(\mathbb{R}_*)$, and also of the Banach space of bounded Radon measures $\mathcal{M}_b(\mathbb{R}_*)$. In case we consider this set equipped with the subspace topology from the weak*-topology on $\mathcal{M}_b(\mathbb{R}_*)$, we can show that even more is true.

Remark. Before proving the next claim, we recall that since $C_0(\mathbb{R}_*)^*$ is isometrically isomorphic to $\mathcal{M}_b(\mathbb{R}_*)$, it follows that the weak*-topology on $\mathcal{M}_b(\mathbb{R}_*) \cong C_0(\mathbb{R}_*)^*$ is given by the seminorms $\{p_\phi \colon \phi \in C_0(\mathbb{R}_*)\}$, where $p_\phi(\mu) = |\langle \phi, \mu \rangle| = \left| \int_{\mathbb{R}_*} \phi(x)\mu(\mathrm{d}x) \right|$.

Lemma 5.2. *The set $\mathscr{X}_L \subset \mathcal{M}_{+,b}(\mathbb{R}_*) \subset \mathcal{M}_b(\mathbb{R}_*)$ is closed in the weak*-topology on $\mathcal{M}_b(\mathbb{R}_*)$.*

Proof. Denote $A = A_1 \cup A_2$, where $A_1 = (0,1)$ and $A_2 = (2L, +\infty)$. For each $\phi \in C_0(\mathbb{R}_*)$, supported in A, define $\Lambda_\phi \colon \mathcal{M}_b(\mathbb{R}_*) \to \mathbb{R}$ by

$$\Lambda_\phi(\mu) = \int_{\mathbb{R}_*} \phi(x)\mu(\mathrm{d}x).$$

Clearly each $\Lambda_\phi \colon \mathcal{M}_b(\mathbb{R}_*) \to \mathbb{R}$ is a continuous mapping, when the domain is equipped with the weak*-topology, since $|\Lambda_\phi(\mu)| \leq |\langle \phi, \mu \rangle|$.

Now, suppose that $\mu(A) = 0$. Then clearly $\Lambda_\phi(\mu) = 0$ for every $\phi \in C_0(\mathbb{R}_*)$, especially for those supported in A.

On the other hand, suppose that $\mu(A) \neq 0$. Since μ is a Radon measure and A is an open set, there exists a non-negative, continuous function $\phi \in C_0(\mathbb{R}_*)$, supported in A, which verifies the inequality $\mu(A) - \mu(A)/2 = \mu(A)/2 < \int_{\mathbb{R}_*} \phi(x)\mu(\mathrm{d}x)$. Therefore, $\Lambda_\phi(\mu) \neq 0$ for some $\phi \in C_0(\mathbb{R}_*)$.

We have shown that $\Lambda_\phi(\mu) = 0$ for every $\phi \in C_0(\mathbb{R}_*)$ that is supported in A if and only if $\mu(A) = 0$. It follows that $\mathscr{X}_L$ can be expressed as

$$\mathscr{X}_L = \mathcal{M}_{+,b}(\mathbb{R}_*) \cap \bigcap \{\Lambda_\phi^{-1}(\{0\}) \colon \phi \in C_0(\mathbb{R}_*), \mathrm{supp}(\phi) \subset A\}.$$

This is a closed set. First of all, the intersection is an arbitrary intersection of closed sets $\Lambda_\phi^{-1}(\{0\})$, these being preimages of closed sets under weak*-continuous mapping. On the other hand, if we define $p_\phi^+(\eta) = \langle \eta, \phi \rangle$ and $C_0^+(\mathbb{R}_*)$ to be the collection of all non-negative $C_0(\mathbb{R}_*)$-functions, we can write

$$\mathcal{M}_{+,b}(\mathbb{R}_*) = \bigcap_{\phi \in C_0^+(\mathbb{R}_*)} (p_\phi^+)^{-1}([0, +\infty))$$

is weak*-closed. Therefore, $\mathscr{X}_L$ is closed as an intersection of two closed sets. □

Lemma 5.3. *The set* $\mathcal{K} \coloneqq \mathscr{X}_L \cap \{f \in \mathcal{M}_b(\mathbb{R}_*)\colon \|f\| \leq 1\}$ *is metrizable in the weak*-topology.*

Proof. By Lemma 5.2, $\mathcal{K}$ is closed as an intersection of two weak*-closed sets. By Banach-Alaoglu's theorem, the set

$$\{f \in \mathcal{M}_b(\mathbb{R}_*)\colon \|f\| \leq 1\}$$

is weak*-compact, and consequently $\mathcal{K}$ is weak*-compact as a closed subset of a compact set. Since $\mathbb{R}_*$ is locally compact, $C_c(\mathbb{R}_*)$ is separable and hence it follows by [27, thm 3.17, p. 70] that $\mathcal{K}$ is metrizable in the weak*-topology. □

Next, we will show that if we define $\mathcal{F}$ by restricting the mapping defined by (5.2) to the set $C([0,T],\mathscr{X}_L) \subset C([0,T],\mathcal{M}_{+,b}(\mathbb{R}_*))$, then we have a mapping $\mathcal{F}\colon C([0,T],\mathscr{X}_L) \to C([0,T],\mathscr{X}_L)$, which is contractive for small enough T.

Lemma 5.4. *For every sufficiently small time length* $T > 0$*, the mapping* $\mathcal{F}$ *given by* (5.2) *maps the set* $C([0,T],\mathscr{X}_L)$ *of continuous, measure-valued functions into itself contractively*[8]*. In particular, we can pick any* $T \leq \frac{C(a_2,\eta)}{1+\|f_0\|}$ *for a suitably picked constant* $C(a_2,\eta)$ *depending on the upper bound of the kernel* K *and the norm of the injection measure.*

Proof. Let $f \in C([0,T],\mathscr{X}_L)$ be given. Let us denote by μ_F the measure-valued function arising from $\mathcal{F}[f]$ by the one-to-one correspondence between $C_0(\mathbb{R}_*)^*$ and $\mathcal{M}_b(\mathbb{R}_*)$. We want to show that the measure-valued function given by the functional valued function $\mathcal{F}[f]$ belongs to the same space, provided that we select a small enough time lenght $T > 0$. In addition, we want to show that the mapping $\mathcal{F}$ is then contractive, which means that

$$\|\mathcal{F}[f] - \mathcal{F}[g]\|_T \leq k\|f - g\|_T. \tag{5.6}$$

Here $k \in [0,1)$ is the so-called contraction constant.

This proof is quite long, so we divide it to a few steps. In the first step, we verify that the for every $t \in [0,T]$, the measure associated with $\mathcal{F}[f](t)$ lies inside $\mathscr{X}_L$ for every $f \in C([0,T],\mathscr{X}_L)$. After this, we show that $\mathcal{F}[f](t)$ varies continuously in $t \in [0,T]$. Finally, we construct a non-empty and complete subset of $C([0,T],\mathscr{X}_L)$ on which $\mathcal{F}$ operates contractively.

[8]This means that the mapping shrinks the metric, which in this case is induced by the operator norm on $C_0(\mathbb{R}_*)^*$, by a contraction constant $k \in [0,1)$.

Step 1 We showed in Lemma 5.1 that $\mu_F(t) \in \mathcal{M}_{+,b}(\mathbb{R}_*)$ for every $t \in [0,T]$, where T can be chosen freely (and later we will choose it to be small enough). Next, we want to show that $\mu_F(t) \in \mathscr{X}_L$, i.e. $\mu_F(t)(A) = 0$, where $A = (0,1) \cup (2,\infty)$.

Let $A_1 = (0,1)$ and $A_2 = (2,\infty)$. We can show that $\mu_F(t)(A_1) = 0$ and $\mu_F(t)(A_2) = 0$, whereby the claim follows.

Trying to arrive at a contradiction, let us assume that for some constant $c \in \mathbb{R}$, $c > 0$, we have

$$\mu_F(t)(A_1) \geq c. \tag{5.7}$$

In other words, we suppose that A_1 is of strictly positive measure with respect to the measure $\mu_F(t)$.

Since $\mu_F(t)$ is a positive, bounded Radon measure, and since A_1 is a Borel set in $\mathbb{R}_*$ with the standard topology, we can find a compact set $K \subset A_1$ such that

$$\mu_F(t)(K) \geq \mu_F(t)(K) - \frac{c}{2} \geq \frac{c}{2}.$$

The sets K and $[1, 2L]$ are both compact and disjoint. Therefore, we can find a positive, continuous cut-off function $\phi \in C_0(\mathbb{R}_*)$ which takes the value 1 on K, vanishes on $[1, 2L]$, and satisfies $\|\phi\| \leq 1$.

Since ϕ is a positive function and $\mu_F(t)$ is a positive measure, we have

$$\begin{aligned}
\mu_F(t)(K) &= \int_{\mathbb{R}_*} \chi_K \mu_F(\mathrm{d}x) \\
&\leq \int_{\mathbb{R}_*} \phi \mu_F(\mathrm{d}x) \\
&= \langle \phi, \mathcal{F}[f](t) \rangle \\
&= \sum_{i=1}^{3} \langle \phi, \mathcal{F}_i[f](t) \rangle = 0.
\end{aligned}$$

Here we have used the fact that the integrals $\langle \phi, \mathcal{F}_i[f](t) \rangle$ vanish. The reason for this is that the integrals are taken over measures that vanish outside the set $[1, 2L]$, and on the other hand ϕ is not supported inside the set $[1, 2L]$. Thus, each integral $\langle \phi, \mathcal{F}[f](t) \rangle$ decomposes into integrals over sets $[1, 2L]$ and $[1, 2L]^c$, which both vanish.

Thus, we have derived that $0 = \mu_F(t)(K) \geq c/2 > 0$. This is a contradiction. Therefore, (5.7) is not true. In particular, $\mu_F(A_1) = 0$, which we wished to show. That the other set A_2 is of measure zero with respect to $\mu_F(t)$ is shown by an analogous argument.

Step 2 The next task is to prove that the measure-valued function $\mathcal{F}[f]$ is continuous as a map from $[0,T]$ to $\mathcal{M}_{+,b}(\mathbb{R}_+)$, when the target set is equipped with the topology inherited from the total variation norm on the space of bounded signed measures $\mathcal{M}_b(\mathbb{R}_+)$. If we can show this, then certainly $\mathcal{F}[f] \in C([0,T], \mathscr{X}_L)$.

To this end, we need to show that we can make $\|\mu_F(t_2) - \mu_F(t_1)\|$ as small as we wish by making the absolute value $|t_2 - t_1|$ sufficiently small. But since the total variation norm of the measure is the same as the operator norm of the corresponding functional, it is enough to show that $\|\mathcal{F}[t](t_2) - \mathcal{F}[f](t_1)\|$ can be made small.

Let $t_2, t_1 \in [0,T]$ be two distinct points. We suppose further, without loss in generality, that $t_2 > t_1$. Moreover, let $\phi \in C_c(\mathbb{R}_*) \subset C_0(\mathbb{R}_*)$ be a continuous, non-negative function in the unit ball of $C_c(\mathbb{R}_*)$, i.e. satisfying the estimate $\|\phi\|_\infty \leq 1$. We can assume this, since we are interested in the norm of the linear functional $\mathcal{F}[f](t_2) - \mathcal{F}[f](t_1)$, which is the operator norm of $(C_0(\mathbb{R}_*))^*$. From the unique extension of densely defined bounded linear operator theorem, it follows that it suffices to consider the values $|\langle \phi, \mathcal{F}[f](t_2) - \mathcal{F}[f](t_1)\rangle|$ for $\phi \in C_c(\mathbb{R}_*)$, since we have a unique extension to a bounded operator on $C_0(\mathbb{R}_*)$ with the same operator norm.

The triangle inequality yields

$$|\langle \phi, \mathcal{F}[f](t_2) - \mathcal{F}[f](t_1)\rangle| \leq \sum_{i=1}^{3} |\langle \phi, \mathcal{F}_i[f](t_2)\rangle - \langle \phi, \mathcal{F}_i[f](t_1)\rangle|. \tag{5.8}$$

To attain a bound for the expression on the left hand side (5.8), we need to find bounds on each of the terms on the right hand side. First,

$$\begin{aligned}
|\langle \phi, \mathcal{F}_1[f](t_2)\rangle - \langle \phi, \mathcal{F}_1[f](t_1)\rangle| &= \left| \int_{\mathbb{R}_*} e^{-\int_0^{t_2} a[f](x,s)\,\mathrm{d}s} \phi(x) f_0(\mathrm{d}x) - \int_{\mathbb{R}_*} e^{-\int_0^{t_1} a[f](x,s)\,\mathrm{d}s} \phi(x) f_0(\mathrm{d}x) \right| \\
&\leq \int_{\mathbb{R}_*} \left| \int_0^{t_2} a[f](x,s)\,\mathrm{d}s - \int_0^{t_1} a[f](x,s)\,\mathrm{d}s \right| f_0(\mathrm{d}x) \\
&= \int_{\mathbb{R}_*} \left| \int_{t_1}^{t_2} a[f](x,s)\,\mathrm{d}s \right| f_0(\mathrm{d}x) \\
&\leq (t_2 - t_1) a_2 \|f_0\| \|f\|_T.
\end{aligned}$$

The expression on the right hand side of the last line can be made as small as we wish by requiring t_2 to be sufficiently close to t_1.

Similarly, we can bound

$$|\langle \phi, \mathcal{F}_3[f](t_2)\rangle - \langle \phi, \mathcal{F}_3[f](t_1)\rangle| \leq (t_2 - t_1)(a_2\|f\|_T + 1)\|\eta\|.$$

The right hand side can be made as small as we wish by letting t_2 and t_1 be close to one another.

Lastly, we have the following estimate

$$\begin{aligned}
&|\langle \phi, \mathcal{F}_2[f](t_2)\rangle - \langle \phi, \mathcal{F}_2[f](t_1)\rangle| \\
\leq &\left| \int_{t_1}^{t_2} \int_{\mathbb{R}_*} f(\mathrm{d}y, s) \int_{\mathbb{R}_*} K(x+y)\phi(x+y) \frac{\zeta_L(x+y)}{2} e^{-\int_s^{t_2} a[f](x+y,\xi)\,\mathrm{d}\xi} f(\mathrm{d}x, s)\,\mathrm{d}s \right| \\
+ &\left| \int_0^{t_1} \int_{\mathbb{R}_*} f(\mathrm{d}y, s) \int_{\mathbb{R}_*} K(x,y)\phi(x+y) \frac{\zeta_L(x+y)}{2} \left(e^{-\int_s^{t_2} a[f](x+y,\xi)\,\mathrm{d}\xi} - e^{-\int_s^{t_1} a[f](x+y,\xi)\,\mathrm{d}\xi} \right) f(\mathrm{d}x, s)\mathrm{d}s \right| \\
\leq &(t_2 - t_1)\frac{a_2}{2}\|f\|_T^2 + T\|f\|_T^2 \frac{a_2}{2}(t_2 - t_1)a_2\|f\|_T.
\end{aligned}$$

Therefore we can bound the norm of the difference $\mu_F(t_2) - \mu_F(t_1)$:

$$\|\mu_F(t_2) - \mu_F(t_1)\| = \|\mathcal{F}[f](t_2) - \mathcal{F}[f](t_1)\| \leq C(t_2 - t_1),$$

where C is a constant depending on the measures η, f_0, the time length T, the supremum norm $\|f\|_T$ and the kernel K. Consequently, the measure-valued function μ_F is continuous in $t \in [0, T]$, i.e.

$$\mu_F \in C([0, T], \mathscr{X}_L).$$

Step 3 Next, let

$$X_T \coloneqq \{f \in C([0, T], \mathscr{X}_L) \colon \|f - f_0\|_T \leq 1 + \|f_0\|, f \geq 0\},$$

where we think of f_0 as the constant function $f_0(t) = f_0$. We would like to show that for a suitably picked time length $T > 0$, the mapping $\mathcal{F} \colon X_T \to X_T$ is a contractive map. To this end, let $f, g \in X_T$. The triangle inequality then gives upper bound on their norms. More precisely,

$$\|f\|_T \leq \|f - f_0\| + \|f_0\|_T \leq 1 + 2\|f_0\|_T,$$

and similarly $\|g\|_T \leq 1 + 2\|f_0\|_T$.
Let $\phi \in C_c(\mathbb{R}_*)$ with $\|\phi\| \leq 1$. Then

$$\begin{aligned}
&|\langle \phi, \mathcal{F}_1[f](t)\rangle - \langle \phi, \mathcal{F}_1[g](t)\rangle| \\
&= \left| \int_{\mathbb{R}_*} \phi(x) e^{-\int_0^t a[f](x,s)\,\mathrm{d}s} f_0(\mathrm{d}x) - \int_{\mathbb{R}_*} \phi(x) e^{-\int_0^t a[g](x,s)\,\mathrm{d}s} f_0(\mathrm{d}x) \right| \\
&\leq \int_{\mathbb{R}_*} \left| \int_0^t a[f](x,s) - a[g](x,s)\,\mathrm{d}s \right| f_0(\mathrm{d}x) \\
&\leq \int_{\mathbb{R}_*} \int_0^t a_2 \left(\int_{\mathbb{R}_*} |f(\mathrm{d}y, s) - g(\mathrm{d}y, s)| \right) \mathrm{d}s f_0(\mathrm{d}x) \\
&\leq a_2 \int_{\mathbb{R}_*} \int_0^t \|f - g\|_T \,\mathrm{d}s f_0(\mathrm{d}x) \\
&\leq a_2 \|f_0\| T \|f - g\|_T \\
&\leq \alpha \|f - g\|_T,
\end{aligned}$$

if $T \leq \frac{\alpha}{a_2} (1 + \|f_0\|)^{-1}$.
Also,

$$\begin{aligned}
&|\langle \phi, \mathcal{F}_3[f](t)\rangle - \langle \phi, \mathcal{F}_3[g](t)\rangle| \\
&= \left| \int_{\mathbb{R}_*} \eta(x)\phi(x) \int_0^t \left(e^{-\int_s^t a[f](x,\xi)\,\mathrm{d}\xi} - e^{-\int_s^t a[g](x,\xi)\,\mathrm{d}\xi} \right) \mathrm{d}s\,\mathrm{d}x \right| \\
&\leq \int_{\mathbb{R}_*} \eta(x) \left| \int_0^t \left(e^{-\int_s^t a[f](x,\xi)\,\mathrm{d}\xi} - e^{-\int_s^t a[g](x,\xi)\,\mathrm{d}\xi} \right) \mathrm{d}s \right| \mathrm{d}x \\
&\leq \int_{\mathbb{R}_*} \eta(x) \int_0^t \left| \int_s^t a[f](x,\xi) - a[g](x,\xi)\,\mathrm{d}\xi \right| \mathrm{d}s\,\mathrm{d}x \\
&\leq a_2 \int_{\mathbb{R}_*} \eta(x) \int_0^t \int_s^t \int_{\mathbb{R}_*} |f(\mathrm{d}y, \xi) - g(\mathrm{d}y, \xi)|\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}x \\
&\leq a_2 \|\eta\| T^2 \|f - g\|_T \\
&\leq \alpha \|f - g\|_T,
\end{aligned}$$

when $T \leq \sqrt{\frac{\alpha}{a_2\|\eta\|}}$ is satisfied.

If we write $h(x;s,t,f) = \frac{\zeta_L(x)}{2}\phi(x)e^{-\int_s^t a[f](x,\xi)\,\mathrm{d}\xi}$

$$
\begin{aligned}
&\Big| \int_0^t \int_{\mathbb{R}_*} f(\mathrm{d}y,s) \int_{\mathbb{R}_*} K(x,y) f(\mathrm{d}x,s) h(x+y;s,t,f)\,\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{R}_*} g(\mathrm{d}y,s) \int_{\mathbb{R}_*} K(x,y) g(\mathrm{d}x,s) h(x+y;s,t,g)\,\mathrm{d}s \Big| \\
&\le \Big| \int_0^t \int_{\mathbb{R}_*} f(\mathrm{d}y,s) \int_{\mathbb{R}_*} K(x,y) f(\mathrm{d}x,s) (h(x+y;s,t,f) - h(x+y;s,t,g))\,\mathrm{d}s \Big| \\
&+ \Big| \int_0^t \int_{\mathbb{R}_*} (f-g)(\mathrm{d}y,s) \int_{\mathbb{R}_*} K(x,y) f(\mathrm{d}x,s) h(x+y;s,t,g)\,\mathrm{d}s \Big| \\
&+ \Big| \int_0^t \int_{\mathbb{R}_*} g(\mathrm{d}y,s) \int_{\mathbb{R}_*} K(x,y) (f-g)(\mathrm{d}x,s) h(x+y;s,t,g)\,\mathrm{d}s \Big| \\
&= J_1 + J_2 + J_3.
\end{aligned}
$$

We want to find useful upper bounds for the terms J_1, J_2 and J_3. For the first term, we calculate

$$
\begin{aligned}
J_1 &= \Big| \int_0^t \int_{\mathbb{R}_*} f(\mathrm{d}y,s) \int_{\mathbb{R}_*} K(x,y) f(\mathrm{d}x,s) (h(x+y;s,t,f) - h(x+y;s,t,g))\,\mathrm{d}s \Big| \\
&\le \frac{a_2^2}{2} \int_0^t \int_{\mathbb{R}_*} f(\mathrm{d}y,s) \int_{\mathbb{R}_*} f(\mathrm{d}x,s) \int_s^t \int_{\mathbb{R}_*} |(f-g)(\mathrm{d}z,\xi)|\,\mathrm{d}\xi\,\mathrm{d}s \\
&\le \frac{a_2^2}{2} T^2 (1 + 2\|f_0\|)^2 \|f-g\|_T.
\end{aligned}
$$

The right hand side can be made smaller than, say, $\alpha\|f-g\|_T$ by selecting the time length T to satisfy $T \le \frac{\sqrt{2\alpha}}{a_2}(1+2\|f_0\|)^{-1}$.

Next, we estimate J_2:

$$
\begin{aligned}
J_2 &= \Big| \int_0^t \int_{\mathbb{R}_*} (f-g)(\mathrm{d}y,s) \int_{\mathbb{R}_*} K(x,y) f(\mathrm{d}x,s) h(x+y;s,t,g)\,\mathrm{d}s \Big| \\
&\le a_2 \int_0^t \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} |(f-g)(\mathrm{d}y,s)| f(\mathrm{d}x,s)\,\mathrm{d}s \\
&\le a_2 T \|f-g\|_T \|f\|_T \\
&\le a_2 T \|f-g\|_T (1 + 2\|f_0\|).
\end{aligned}
$$

Once again, the right hand side on the last line can be made smaller than $\alpha\|f-g\|_T$ by requiring T to satisfy $T \le \frac{\alpha}{a_2}(1+2\|f_0\|)^{-1}$.

Finally,

$$\begin{aligned} J_3 &= \Big| \int_0^t \int_{\mathbb{R}_*} g(\mathrm{d}y, s) \int_{\mathbb{R}_*} K(x,y)(f-g)(\mathrm{d}x, s) h(x+y; s, t, g)\, \mathrm{d}s \Big| \\ &\leq a_2 \int_0^t \int_{\mathbb{R}_*} g(\mathrm{d}y, s) \int_{\mathbb{R}_*} |f-g|\, \mathrm{d}s \\ &\leq a_2 T \|g\|_T \|f-g\|_T \\ &\leq a_2 T \|f-g\|_T (1 + 2\|f_0\|) \end{aligned}$$

This can be bounded by $\alpha \|f-g\|_T$ by selecting $T \leq \frac{\alpha}{a_2}(1+2\|f_0\|)^{-1}$

What is left to show that the image of the map $\mathcal{F}$ is a subset of X_T. In other words, we would like to show that for a suitably selected T, we have

$$\|\mathcal{F}[f] - f_0\|_T \leq 1 + \|f_0\| \tag{5.9}$$

If we select $\alpha > 0$ be small enough, the contractivity of the operator $\mathcal{F}[f](t)$ follows with a contractive constant $k = 5\alpha < 1$. By the triangle inequality, we have

$$\|\mathcal{F}[f] - f_0\|_T \leq \|\mathcal{F}[f] - \mathcal{F}[f_0]\|_T + \|\mathcal{F}[f_0] - f_0\|_T \leq k(1 + \|f_0\|) + \|\mathcal{F}[f_0] - f_0\|_T.$$

The k-estimate is obvious, since $\mathcal{F}$ is contractive and $\|f - f_0\|_T \leq 1 + \|f_0\|$ for $f \in X_T$. Inequality (5.9) holds, if we manage to show that $\|\mathcal{F}[f_0] - f_0\|_T < \gamma(1 + \|f_0\|)$, where $\gamma > 0$ is a constant satisfying

$$\gamma < (1-k).$$

To this end, we need to make some additional estimates on how small T should be.

We can make the expression

$$\left| \int_0^t \int_{\mathbb{R}_*} f_0(\mathrm{d}y, s) \int_{\mathbb{R}_*} K(x,y) f_0(\mathrm{d}x, s) h(x+y; s, t, f_0)\, \mathrm{d}s \right|$$

smaller than, say, $\frac{\gamma}{5}$, if we require T to be smaller than $\frac{C}{a_2}\,(1+2\|f_0\|)^{-1}$ for some absolute constant C.

Similarly, we can make the expression

$$\left| \int_{\mathbb{R}_*} \eta(\mathrm{d}x) \int_0^t e^{-\int_s^t a[f_0](x,\xi)\, \mathrm{d}\xi}\, \mathrm{d}s \right|$$

smaller than $\frac{\gamma}{5}$, in case we require $T \leq \frac{C}{10\|\eta\|}$ for some absolute constant C. Finally, we can bound

$$\left| \int_{\mathbb{R}_*} f_0(\mathrm{d}x) - \int_{\mathbb{R}_*} e^{-\int_0^t a[f_0](x,s)\,\mathrm{d}s} f_0(\mathrm{d}x) \right|$$

by $\frac{\gamma}{5}(1 + \|f_0\|)$ if we let $T \leq \frac{C(a_2)}{(1+2\|f_0\|)}$.
Combining these fact, $\mathcal{F}$ maps $X_T \to X_T$ contractively, provided that we let

$$T \leq \min\left(\frac{C_1(a_2)}{(1+2\|f_0\|)}, C_2(a_2, \eta) \right).$$

The fact that $\mathcal{F}$ maps X_T to X_T comes from the estimate

$$\|\mathcal{F}[f] - f_0\| \leq (k + \gamma)(1 + \|f_0\|) \leq 1 + \|f_0\|.$$

We note that the conditions posed on T in the above proof can be captured by a single condition. Indeed, the claim that $\mathcal{F}\colon X_T \to X_T$ contractively holds, in case

$$T \leq \frac{C'}{1 + 2\|f_0\|}.$$

For a suitably picked constant $C' = C'(a_2, \eta)$. In addition, note that we can replace the constant C' by $C'/2$ to get contractivity assuming that

$$T \leq \frac{C}{1 + \|f_0\|},$$

since then also

$$T \leq \frac{C}{1 + \|f_0\|} \leq \frac{C'}{2 + 2\|f_0\|} \leq \frac{C'}{1 + \|f_0\|}.$$

Importantly, $C = C(a_2, \eta) = C'(a_2, \eta)/2$ does not depend on $\|f_0\|$. □

Having established the contractivity of the operator $\mathcal{F}\colon X_T \to X_T$, we can use this to find a unique fixed point to it. This in turn can be made into an argument for the existence of a solution to the coagulation equation with injection.

Lemma 5.5. *Given small enough $T \in \mathbb{R}_+$, there exists a unique measure-valued function $f \in X_T$ which is a fixed point of $\mathcal{F}$. In other words, we have $f = \mathcal{F}[f]$.*

Proof. X_T is a complete metric space as a closed subset of the complete metric space $C([0,T],\mathscr{X}_L)$. Moreover, $\mathcal{F}$ acts contractively on this space. Thus, by Banach's fixed point theorem, there exists a unique fixed point $f \in X_T$. □

Next, we want to show that the fixed point f of $\mathcal{F}$ is in fact a function in $C^1([0,T],\mathscr{X}_L)$, as defined in

Lemma 5.6. *The fixed point $f = \mathcal{F}[f]$ lies inside $C^1([0,T],\mathcal{M}_{+,b}(\mathbb{R}_*))$ and solves the equation*

$$\begin{aligned}\int_{\mathbb{R}_*} \phi(x,t)\dot{f}(\mathrm{d}x,t) = \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\left(\phi(x+y)\zeta_L(x+y) - \phi(x,t) - \phi(y,t)\right) f(\mathrm{d}x,t)f(\mathrm{d}y,t) \\ + \int_{\mathbb{R}_*} \phi(x,t)\eta(\mathrm{d}x) \end{aligned} \tag{5.10}$$

for every test function $\phi \in C^1([0,T],C_c(\mathbb{R}_*))$.

Thereby, it is a solution to the time-dependent regularized coagulation equation in the sense of Definition 4.3.

Proof. First, we want to verify that

$$f \in C^1([0,T],\mathcal{M}_{+,b}(\mathbb{R}_*)).$$

We will do this by presenting a measure-valued function that satisfies the definition of the Fréchet derivative of f. Indeed, we want to show that the bounded Radon measure (not necessarily positive) valued function $g\colon [0,T] \to \mathcal{M}_b(\mathbb{R}_*)$ corresponding to the functional

$$\begin{aligned}\int_{\mathbb{R}_*} g(\mathrm{d}x,t)\phi(x) = \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\left(\zeta_L(x+y)\phi(x+y) - \phi(x) - \phi(y)\right) f(\mathrm{d}x,t)f(\mathrm{d}y,t) \\ + \int_{\mathbb{R}_*} \phi(x)\eta(\mathrm{d}x)\end{aligned}$$

is actually $\dot{f}$.

This holds, in case we can show that for every $t \in (0,T)$, we have

$$\frac{\|f(t+h) - f(t) - g(t)h\|}{|h|} \to 0 \text{ as } h \to 0. \tag{5.11}$$

We note that this happens in case $(f(t+h)-f(t))h^{-1} \to g(t)$ as $h \to 0$ in the norm topology.

Indeed, for any given $\phi \in C_c(\mathbb{R}_*)$ with $\|\phi\| = 1$, we have

$$\begin{aligned}\langle \phi, (f(t+h)-f(t))\rangle &= \langle \phi, \mathcal{F}[f](t+h) - \mathcal{F}[f](t)\rangle \\ &= \sum_{i=1}^{3} \left(\langle \phi, \mathcal{F}_i[f](t+h)\rangle - \langle \phi, \mathcal{F}_i[f](t)\rangle\right).\end{aligned}$$

We want to study each of the terms in the sum. First of all, we have

$$\frac{\left(\langle \phi, \mathcal{F}_1[f](t+h)\rangle - \langle \phi, \mathcal{F}_1[f](t)\rangle\right)}{h} = \left(\int_{R_*} \phi(x)\left(e^{-\int_0^{t+h} a[f](x,s)\,\mathrm{d}s} - e^{-\int_0^t a[f](x,s)\,\mathrm{d}s}\right) f_0(\mathrm{d}x)\right) h^{-1}.$$

By dominated convergence, the expression on the right hand side converges to

$$-\langle a[f](\cdot,t)\phi, \mathcal{F}_1[f](t)\rangle \text{ as } h \to 0.$$

Similarly, we have

$$\begin{aligned}&\left(\langle \phi, \mathcal{F}_3[f](t+h)\rangle - \langle \phi, \mathcal{F}_3[f](t)\rangle\right) h^{-1} \\ &= \int_{\mathbb{R}_*} \phi(x) h^{-1} \left(\int_0^{t+h} e^{-\int_s^{t+h} a[f](x,\xi)\,\mathrm{d}\xi}\,\mathrm{d}s - \int_0^t e^{-\int_s^t a[f](x,\xi)\,\mathrm{d}\xi}\,\mathrm{d}s\right)\eta(\mathrm{d}x)\end{aligned}$$

Using dominated convergence, we can show that this converges to

$$\int_{\mathbb{R}_*} \phi(x)\eta(\mathrm{d}x) - \langle a[f](\cdot,t)\phi, \mathcal{F}_3[f](t)\rangle$$

Finally, once again using dominated convergence, we have

$$\begin{aligned}&\left(\langle \phi, \mathcal{F}_2[f](t+h)\rangle - \langle \phi, \mathcal{F}_2[f](t)\rangle\right) h^{-1} \\ &\to \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \phi(x+y)K(x,y)\zeta_L(x+y) f(\mathrm{d}x,t) f(\mathrm{d}y,t) - \langle a[f](\cdot,t)\phi, \mathcal{F}_2[f](t)\rangle.\end{aligned}$$

Moreover, we note that the above convergences are uniform in $\|\phi\| \le 1$.

Therefore,

$$
\begin{aligned}
&\left(\langle\phi, \mathcal{F}[f](t+h)\rangle - \langle\phi, \mathcal{F}[f](t)\rangle\right) h^{-1} \\
&\to \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \phi(x+y)K(x,y)\zeta_L(x+y)f(\mathrm{d}x,t)f(\mathrm{d}y,t) - \langle a[f](\cdot,t)\phi, \mathcal{F}[f](t)\rangle + \int_{\mathbb{R}_*}\phi(x)\eta(\mathrm{d}x) \\
&= \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \phi(x+y)K(x,y)\zeta_L(x+y)f(\mathrm{d}x,t)f(\mathrm{d}y,t) - \langle a[f](\cdot,t)\phi, f(t)\rangle + \int_{\mathbb{R}_*}\phi(x)\eta(\mathrm{d}x) \\
&= \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \phi(x+y)K(x,y)\zeta_L(x+y)f(\mathrm{d}x,t)f(\mathrm{d}y,t) - \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\phi(x)f(\mathrm{d}x,t)f(\mathrm{d}y,t) \\
&- \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \phi(y)K(x,y)f(\mathrm{d}x,t)f(\mathrm{d}y,t) + \int_{\mathbb{R}_*}\phi(x)\eta(\mathrm{d}x) \\
&= \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\left[\zeta_L(x+y)\phi(x+y) - \phi(x) - \phi(y)\right] f(\mathrm{d}x,t)f(\mathrm{d}y,t) + \int_{\mathbb{R}_*}\phi(x)\eta(\mathrm{d}x)
\end{aligned}
$$

Uniformly for $\phi \in C_c(\mathbb{R}_*)$ with $\|\phi\| \leq 1$. Therefore, in the operator norm topology, for every $t \in (0,T)$, we have

$$\frac{f(t+h) - f(t)}{h} \to g(t), \text{ as } h \to 0.$$

Moreover, since $g(t)$ can be continuously extended to the whole interval $[0,T]$, we can extend $\dot{f}$ to the two end points 0 and T. Therefore, $f \in C^1([0,T], \mathcal{M}_{+,b}(\mathbb{R}_*))$, and its derivative acts as the functional on $C_c(\mathbb{R}_*)$ given by

$$\left\langle \phi, \dot{f}(t)\right\rangle = \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\left[\zeta_L(x+y)\phi(x+y) - \phi(x) - \phi(y)\right] f(\mathrm{d}x,t)f(\mathrm{d}y,t) + \int_{\mathbb{R}_*}\phi(x)\eta(\mathrm{d}x).$$

Since $f \in C^1([0,T], \mathcal{M}_{+,b}(\mathbb{R}_*))$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_*}\phi(x,t)f(\mathrm{d}x,t) = \int_{\mathbb{R}_*}\dot{f}(\mathrm{d}x,t)\phi(x,t) + \int_{\mathbb{R}_*}\dot{\phi}(x,t)f(\mathrm{d}x,t) \tag{5.12}$$

In particular, we note that for any $\phi \in C^1([0,T], C_c(\mathbb{R}_*))$, the function $\phi(t)$ lies inside $C_c(\mathbb{R}_*)$. This verifies (5.10). □

Remark. It follows that for every $t \in [0,T]$, we have

(5.13)
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}_*} f(\mathrm{d}x,t)\right) \leq \frac{1}{2}\int_{\mathbb{R}_*} f(\mathrm{d}y,t)\int_{\mathbb{R}_*} K(x,y)f(\mathrm{d}x,t) - \int_{\mathbb{R}_*} f(\mathrm{d}y,t)\int_{\mathbb{R}_*} K(x,y)f(\mathrm{d}x,t) + \int_{\mathbb{R}_*}\eta(\mathrm{d}x).$$

Let $t \in (0,T)$. We can find a function $\phi \in C_c(\mathbb{R}_*)$ with the additional properties that $0 \le \phi \le 1$ and $\phi(x) = 1$ on the support of $f(t)$ and η. For this specific function, we have

$$\begin{aligned}\int_{\mathbb{R}_*} \dot{f}(\mathrm{d}x,t)\phi(x) = \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \zeta_L(x+y)K(x,y)\phi(x+y)f(\mathrm{d}x,t)f(\mathrm{d}y,t) \\ - \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\phi(x)f(\mathrm{d}x,t)f(\mathrm{d}y,t) + \int_{\mathbb{R}_*}\phi(x)\eta(\mathrm{d}x)\end{aligned}$$

By positivity, the first integral is certainly smaller than

$$\frac{1}{2}\int_{\mathbb{R}_*} f(\mathrm{d}x,t)\int_{\mathbb{R}_*} K(x,y)f(\mathrm{d}y,t).$$

On the other hand, in the support of $f(t)$, $\phi(x) = 1$, so the integral with the negative sign is identical to

$$\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)f(\mathrm{d}x,t)f(\mathrm{d}y,t).$$

Finally, $\phi(x) = 1$ in the support of η, so

$$\int_{\mathbb{R}_*} \phi(x)\ \eta(\mathrm{d}x) = \int_{\mathbb{R}_*} \eta(\mathrm{d}x).$$

Consequently,

$$\begin{aligned}\int_{\mathbb{R}_*} \dot{f}(\mathrm{d}x,t)\phi(x) \le \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)f(\mathrm{d}x,t)f(\mathrm{d}y,t) \\ - \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)f(\mathrm{d}x,t)f(\mathrm{d}y,t) + \int_{\mathbb{R}_*}\eta(\mathrm{d}x).\end{aligned}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi(t), f(t)\rangle = \left\langle\dot{\phi}(t), f(t)\right\rangle + \left\langle\phi(t), \dot{f}(t)\right\rangle = \left\langle\phi, \dot{f}(t)\right\rangle$, the desired inequality (5.13) follows.

Next, we want to establish some upper bounds on the growth of the measure f in time t. This is the content of the next lemma.

Lemma 5.7. *If $f \in C^1([0,+\infty), \mathcal{M}_{+,b}(\mathbb{R}_*)) \cap C([0,\infty), \mathcal{M}_{+,b}(\mathbb{R}_*))$ is a time-dependent solution in the sense of Definition 4.3, then*

$$\int_{\mathbb{R}_*} f(\mathrm{d}x,t) \le \max\left(\|f_0\|, \|\eta\|\right)(1+t),\ t \ge 0. \tag{5.14}$$

Proof. By Equation (5.13), we have the estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}_*} f(\mathrm{d}x,t)\right) &\leq \frac{1}{2}\int_{\mathbb{R}_*} f(\mathrm{d}y,t)\int_{\mathbb{R}_*} K(x,y)f(\mathrm{d}x,t)\\ &\quad -\int_{\mathbb{R}_*} f(\mathrm{d}y,t)\int_{\mathbb{R}_*} K(x,y)f(\mathrm{d}x,t)+\int_{\mathbb{R}_*}\eta(\mathrm{d}x)\\ &\leq \int_{\mathbb{R}_*}\eta(\mathrm{d}x) \qquad (5.15)\end{aligned}$$

We then integrate both sides from 0 to s with respect to t. The fundamental theorem of calculus gives us

$$\int_{\mathbb{R}_*} f(\mathrm{d}x,s)-\int_{\mathbb{R}_*} f(\mathrm{d}x,0)\leq s\int_{\mathbb{R}_*}\eta(\mathrm{d}x), \qquad (5.16)$$

whereby

$$\int_{\mathbb{R}_*} f(\mathrm{d}x,s)\leq (s+1)\max(\|f_0\|,\|\eta\|).$$

Therefore, the estimate (5.14) is verified. □

The previous lemma is important, because it allows us to control the norm of $f(t)$ in terms of the initial data and the injection measure. It will be particularly useful when we try to establish the existence of solutions of the time-dependent solution for arbitrary times, which we set out to do next. Indeed, up to this point, we have only been able to find time-dependent solution up to some (possibly small) time T, since the smallness of T was used to secure a contractive mapping with a possible fixed point. Now, we want to paste together many small intervals to get a solution that is defined for all points of time $t\in\mathbb{R}_+$.

Proposition 5.8. *Given $f_0\in\mathscr{X}_L$, there exists a solution $f\in C^1([0,+\infty),\mathscr{X}_L)$ of the initial value problem.*

Proof. Our strategy is to show that given an initial value f_0 with $\|f_0\|<+\infty$, we have a unique solution f^1 to the initial value problem in the closed interval $[0,T_0]$ for some $T_0>0$. If we let $f_0^1 \coloneqq f^1(\cdot,T_0/2)$ to be a new initial condition, we can find a unique solution f^2 to the initial value problem in the interval $[0,T_1]$. Again, letting $f_0^2=f^2(\cdot,T_1/2)$ be a new inital value, we get a solution f^3 on the interval $[0,T_3]$. After i steps, we find a solution f^i on the interval $[0,T_i]$ to the initial value problem with the initial data $f^{i-1}(T_{i-1}/2)$. We

can then let $\tilde{f}_i$ to be a measure-values function defined on the interval $[0, \frac{1}{2}\sum_{j=1}^{i-1} T_j + T_i]$ by shifting all solutions f^j, $j < i$ forward by $T_j/2$ and pasting these together to get a solution on the interval $I_i := [0, \sum_{j=1}^{i-1} T_j/2 + T_i]$. Since the interiors of the shifted intervals $[T_i/2, T_i/2 + T_{i+1}]$ cover the whole solution interval of $\tilde{f}_i$, the resulting function $\tilde{f}_i$ is $C^1(I_i, \mathscr{X}_L)$.

Now, $\tilde{f}_i$ is a solution on the interval I_i to the initial value problem with f_0 as the initial value.

Note that $I_1 \subset I_2 \subset \ldots I_j \subset I_{j+1} \subset \ldots$, so these sets form an increasing chain. If we can demonstrate that $\bigcup_{i=1}^{\infty} I_i = [0, +\infty)$, then the existence of a solution on the whole positive real line follows. To do this, it is enough to show that $\sum_{i=0}^{\infty} T_i = +\infty$.

The contractivity works for any $T \leq \frac{C(a_2,\eta)}{1+\|f_0\|}$, for a constant depending on a_2 and the initial measure η. We can choose any T for which the contractivity works. In particular, we can assume that $T = \frac{C(a_2,\eta)}{1+\|f_0\|}$.

At this point, we want to use the inequality,

$$\|f(t)\| \leq \|f_0\| + t\|\eta\|.$$

This follows immediately from (5.16), as it is assumed that $f \in C^1([0,T], \mathcal{M}_{+,b}(\mathbb{R}_*))$.

Since $T_0 \leq \frac{C(a_2,\eta)}{1+\|f_0\|}$, we have $\|f_1\| \leq \|f_0\| + \frac{C(a_2,\eta)\|\eta\|}{1+\|f_0\|}$. Taking f_1 as the new initial data, we have the estimate

$$T_1 \leq \frac{\|C(a_2,\eta)\|}{1+\|f_1\|}.$$

We can select the largest T_1 which satisfies this, so let $T_1 = \frac{C(a_2,\eta)}{1+\|f_1\|}$. But now,

$$T_1 = \frac{C(a_2,\eta)}{1+\|f_1\|} \geq \frac{C(a_2,\eta)}{1+\|f_0\| + \frac{C(a_2,\eta)\|\eta\|}{1+\|f_0\|}}.$$

We can make T_1 as small as we want, provided that we stay above the previous estimate. In particular, we can reselect T_1 in a way that simultaneously guarantees the upper bound

$$T_1 \leq \frac{C(a_2,\eta)}{1+\|f_0\|}.$$

This gives

$$\|f_2\| \leq \|f_0\| + (T_0 + T_1)\|\eta\| \leq \|f_0\| + \frac{2C(a_2,\eta)\|\eta\|}{1+\|f_0\|}$$

Therefore,

$$T_2 \geq \frac{C(a_2,\eta)}{1+\|f_0\|+\frac{2C(a_2,\eta)\|\eta\|}{1+\|f_0\|}}.$$

Once again, we can also guarantee that $T_2 \leq \frac{C(a_2,\eta)}{1+\|f_0\|}$.

Proceeding along these lines, we get the following lower bounds for every $k \in \mathbb{N}$:

$$T_k \geq \frac{C(a_2,\eta)}{1+\|f_0\|+\frac{kC(a_2,\eta)\|\eta\|}{1+\|f_0\|}} \geq \frac{C(a_2,\eta)}{1+\|f_0\|+kC(a_2,\eta)\|\eta\|}.$$

Notice that the coefficients T_k behave asymptotically like the coefficients of the harmonic series, which diverges. This allows us to show that

$$\lim_{N\to\infty}\sum_{k=1}^{N} T_k = \infty.$$

By the remark in the beginning of the proof, this means that the solution exists for arbitrarily long times, i.e. there is a solution $f \in C^1([0,+\infty),\mathscr{X}_L)$. □

Finally, the uniqueness of the solution follows once we prove Proposition 5.9, where it is shown that for any two solutions to the time-dependent initial value problem, any $t \geq 0$ and any $\psi \in C_0(\mathbb{R}_*)$, we can find $\phi \in C_0(\mathbb{R}_*)$ satisfying

$$\int_{\mathbb{R}_*}\psi(x)\left(f(\mathrm{d}x,t)-\widetilde{f}(\mathrm{d}x,t)\right)=\int_{\mathbb{R}_*}\psi(x)\left(f(\mathrm{d}x,0)-\widetilde{f}(\mathrm{d}x,0)\right).$$

Supposing that there are two time-dependent solutions with a common initial value. The above identity then implies that the solutions must be identical.

5.2 From time-dependent solutions to stationary solutions

Once we establish the uniqueness of the time-dependent solution corresponding to an initial value, we can use this to define a semigroup of operators $\{S(t)\}_{t\in\mathbb{R}_+}$ that evolves each initial measure according to the time-dependent solution. More precisely, for each $t \in \mathbb{R}_+$ the operator $S(t)$ acts on the set $\mathscr{X}_L$ by assigning $f_0 \in \mathscr{X}_L$ to $f(t) \in \mathscr{X}_L$, where $f \in C([0,+\infty),\mathscr{X}_L)$ is the unique time-dependent solution corresponding to the initial

measure f_0. These mappings then constitute a semigroup of operators on $\mathscr{X}_L$. In other words, $S(0) = I$, the identity operator on $\mathscr{X}_L$, and $S(t+s) = S(t)S(s)$ for $t, s \geq 0$. The operators in this semigroup are also weak*-continuous as a map on $\mathscr{X}_L$ and the maps from t to $S(t)f_0$ is continuous when the target is equipped with the weak*-topology.

Proposition 5.9. *For a fixed $t \geq 0$, the mapping*

$$\begin{aligned} \mathscr{X}_L &\to \mathscr{X}_L \\ f_0 &\mapsto f(t), \end{aligned}$$

where f is a time-dependent solution corresponding to f_0, is weak-continuous. Moreover, $f(t)$ is uniquely given by f_0, whereby we can define the semigroup of operators $\{S(t)\}_{t\in\mathbb{R}_+}$ which acts on $f_0 \in \mathscr{X}_L$ by $S(t)f_0 = f(t) \in \mathscr{X}_L$.*

Remark. At the moment, we will let $t \in \mathbb{R}_+$. For any $t_0 > 0$, we can find $\alpha \in \mathbb{N}$ and $\beta < t_0$ such that $t = \alpha t_0 + \beta$. The estimate which establishes the weak*-continuity of $f_0 \mapsto f(r)$ for $r \leq t_0$ also shows that any time-dependent solution corresponding to f_0 must be unique up to t_0. Therefore, this allows us to define operators $S(r)$ for $r \leq t_0$. After establishing the weak*-continuity for $f_0 \mapsto f(r)$ for every $r \leq t_0$ then the claim for any $t \in \mathbb{R}_+$ follows from the fact that $f(\cdot, t) = (S(t_0)^\alpha + S(\beta))f_0$. From this, the weak*-continuity of $f_0 \mapsto S(t)f_0$ follows from the facts that a finite composition of continuous functions is continuous and $\mathcal{M}_b(\mathbb{R}_*)$ equipped with the weak*-topology is a topological vector space. This also shows that $f(r)$ is unique up to $r \leq t$. Since $t \in \mathbb{R}_+$ was assumed to be arbitrary, this shows that the time-dependent solution f corresponding to f_0 is unique, whereby we can define the semigroup of operators $\{S(t)\}_{t\in\mathbb{R}_+}$.

Proof. Let f and $\widetilde{f}$ be two measures that are time-dependent solutions to the initial value problems with initial measures $f_0, \widetilde{f}_0$ according to Definition 4.3. We want to show that for a small enough $t_0 \in \mathbb{R}_+$, for every test function $\psi \in C_0(\mathbb{R}_*)$, and for any given $\varepsilon > 0$, we can find a $\delta > 0$ and $\phi \in C_0(\mathbb{R}_*)$ satisfying

$$\left| \int_{\mathbb{R}_*} \psi(x) \left(f(\mathrm{d}x, t_0) - \widetilde{f}(\mathrm{d}x, t_0) \right) \right| < \varepsilon$$

whenever

$$\left| \int_{\mathbb{R}_*} \phi(x) \left(f(\mathrm{d}x, 0) - \widetilde{f}(\mathrm{d}x, 0) \right) \right| < \delta.$$

Since $f \in \mathscr{X}_L$ and $K \in C_c(\mathbb{R}_*^2)$ vanishes for $x, y \leq b$ for some $b > 0$, we only need to consider functions $\psi \in C(A)$, where $A = [b, 4L]$. If this can be shown, it follows that the mapping $f_0 \mapsto f(\cdot, t_0)$ is continuous in the weak*-topology.

Note that we can restrict our attention to times $t_0 < 1$. Therefore, if we fix $T = 1$, we know that $\|f\|_T, \|\widetilde{f}\|_T \leq M < +\infty$.

Let $\psi \in C(A)$ be given. Suppose that we can find a function $\phi^* \in C^1([0, t_0], C(A))$. Since $f_0, \widetilde{f}_0 \in \mathscr{X}_L$ are two initial measures, we can evolve them according to (some) solution to the initial value problem: $f_0 \mapsto f(\cdot, t_0)$ and $\widetilde{f}_0 \mapsto \widetilde{f}(\cdot, t_0)$. Therefore, since f and $\widetilde{f}$ are both time-dependent solutions to the corresponding initial value problems according to Definition 4.3, we have

$$\begin{aligned} &\int_A \phi^*(x,t) \left(f(\mathrm{d}x, t_0) - \tilde{f}(\mathrm{d}x, t) \right) - \int_A \phi^*(x,0) \left(f_0(\mathrm{d}x) - \tilde{f}_0(\mathrm{d}x) \right) \\ &= \int_0^t \int_A \left(f(\mathrm{d}x, t_0) - \tilde{f}(\mathrm{d}x, t_0) \right) \left(\partial_t \phi^*(x,s) + \mathcal{T}[\phi^*](x,s) \right) \mathrm{d}s. \end{aligned} \tag{5.17}$$

For any given function $\phi \in C(A)$, we use the shorthand $\mathcal{T}[\phi]$ to stand for

$$\mathcal{T}[\phi](x,s) \coloneqq \frac{1}{2} \int_A K(x,y) \left(\phi(x+y,s)\zeta_L(x+y) - \phi(x,s) - \phi(y,s) \right) \left(f(\mathrm{d}y, s) + \tilde{f}(\mathrm{d}y, s) \right). \tag{5.18}$$

For a fixed $s \in [0, t_0]$, consider the function $\mathcal{T}[\phi](s)$, which maps x to $\mathcal{T}[\phi](x,s)$. The integrand in $\mathcal{T}[\phi](s)$ is continuous on a compactly supported set, so it is uniformly continuous. These considerations, together with the fact that $(f(s) + \tilde{f}(s))(A) < +\infty$, mean that $\mathcal{T}[\phi](s) \in C(A)$. Moreover, the function $s \mapsto \mathcal{T}[\phi](s)$ is continuous. This follows from the fact that

$$\|f\|_T, \|\widetilde{f}\|_T \leq M < +\infty.$$

Consider the Banach space $C([0, t_0], C(A))$. Let us define

$$\mathscr{A} \coloneqq \{\phi \in C([0, t_0], C(A)) \colon \phi(t) = \psi\}.$$

In other words, the elements of this subset are those in $C([0, t_0], C(A))$ that agree with the previously given function $\psi \in C(A)$ at the endpoint t_0. Eventually, we want to be able to apply Banach's fixed point theorem to find some fixed point of a relevant operator on $\mathscr{A}$, which then allows us to establish the claim we are tying to prove.

Before trying to find a suitable operator, we want to show that $\mathscr{A}$ is a closed set, whereby it is complete. To do this, we first note that the norm topology of $C([0, t_0], C(A))$ is given by the supremum norm

$$\sup_{s \in [0,t_0]} \|g(s)\| = \sup_{s \in [0,t_0]} \sup_{x \in A} |g(x,s)|.$$

Since the norm topology of $C([0,t_0],C(A))$ is metrizable, we can use sequences to show that its subset $\mathscr{A}$ is closed. Indeed, let $\phi_n \in \mathscr{A}$ be a Cauchy sequence. Since $C([0,t_0],C(A))$ is a Banach space, there exists a limit $\phi \in C([0,t_0],C(A))$ such that

$$\|\phi - \phi_n\|_{C([0,t],C(A))} \to 0.$$

Recall that for each ϕ_n, we have $\phi_n(t_0) = \psi$. From the convergence of ϕ_n to ϕ, it follows that

$$\|\phi_n(t_0) - \phi(t_0)\|_{C(A)} = \|\psi - \phi(t_0)\|_{C(A)} \to 0.$$

Therefore at the endpoint t_0, we have $\phi(t_0) = \psi$. In other words, $\phi \in \mathscr{A}$, and $\phi_n \to \phi$ in $C([0,t_0],C(A))$. This establishes that the set $\mathscr{A}$ is closed.

Given a function $\phi \in C([0,\infty),C(A))$, we define an assignment K by letting

$$K\phi = R\phi + P\phi.$$

Here P and S are assignments. Here the map $P\phi$ is given by

$$P\phi(x,s) = \widetilde{\psi}(x,s) = \psi(x),$$

i.e. it is a constant map that sends $\phi \in C([0,t_0],C(A))$ to the element $\widetilde{\psi} \in C([0,t_0],C(A))$ that is defined to be ψ at every point in time. On the other hand, we define R by assigning ϕ to the function $R\phi$, which is given pointwise by

$$R\phi(s) = -\int_s^{t_0} \mathcal{T}[\phi](\cdot,\xi)\,\mathrm{d}\xi \in C(A). \tag{5.19}$$

In case $\phi^* \in C([0,t_0],C(A))$ and satisfies $\phi^* = K\phi^*$, then clearly $\phi^* \in C^1([0,t_0],C(A))$. Therefore, we want to find a fixed point of the operator K.

Moreover, we note that if $\phi \in \mathscr{A}$, then $K\phi(t_0) = R\phi(t_0) + P\phi(t_0) = \psi$. Thus, the restriction of K to $\mathscr{A}$ maps the set to itself. If we can show that K is a contraction on this set, we are done. To this end, recall that we have the following bounds $\|f\|_T, \|\widetilde{f}\|_T \leq M < +\infty$. Therefore, for functions $\phi, \eta \in C([0,t_0],C(A))$, we have

$$\begin{aligned}\sup_{s\in[0,t_0]} \|R\phi(s) - R\eta(s)\| &\leq t_0 \sup_{s\in[0,t_0]} \sup_{x\in A} |\mathcal{T}[\phi](x,s) - \mathcal{T}[\eta](x,s)| \\ &\leq CM \sup_{s\in[0,t_0]} \|\phi - \eta\|,\end{aligned}$$

where $C > 0$ is some absolute constant and M is the aforementioned constant depending on the solutions f and $\widetilde{f}$ via their norms $\|\cdot\|_T$ with $T = 1$.

Therefore, if we let $t_0 > 0$ be smaller than $(2CM)^{-1}$, the restricted operator

$$K|_{\mathscr{A}} \colon \mathscr{A} \to \mathscr{A}$$

is a contraction. By Banach's fixed point theorem there exists a unique fixed point $\phi^* \in \mathscr{A} \subset C([0, t_0], C(A))$ such that $\phi^* = K|_{\mathscr{A}}\phi^*$. We know that $\phi^*(t_0) = \psi$ and $\phi^* \in C^1([0, t_0], C(A))$. Furthermore, the time-derivative function satisfies $\dot{\phi}^*(s, x) = -\mathcal{T}[\phi^*](x, s)$. Applying this identity to equation (5.17) gives us

$$\int_{\mathbb{R}_*} \psi(x) \left(f(\mathrm{d}x, t_0) - \widetilde{f}(\mathrm{d}x, t_0) \right) = \int_{\mathbb{R}_*} \phi(x, 0) \left(f(\mathrm{d}x, 0) - \widetilde{f}(\mathrm{d}x, 0) \right). \tag{5.20}$$

It follows that $f_0 \mapsto S(r)f_0$ is continuous for every $r \leq t_0$. By our Remark at the beginning of the proof, it follows that $f_0 \mapsto S(t)f_0$ is weak*-continuous for every $t \in \mathbb{R}_+$. □

Lemma 5.10. *For a fixed initial measure $f_0 \in \mathscr{X}_L$, the mapping*

$$\begin{aligned} \alpha \colon \mathbb{R}_+ &\to \mathscr{X}_L \\ \alpha(t) &= S(t)f_0 \end{aligned}$$

is continuous, when the domain $\mathbb{R}_+$ is equipped with the usual topology and the image $\mathscr{X}_L$ is equipped with the weak-topology.*

Proof. Let $t_1, t_2 \in [0, T]$, and assume without loss of generality $t_2 > t_1$. Suppose that $\phi \in C_c(\mathbb{R}_*)$. This function is associated with the seminorm $p_\phi(f) = \left| \int_{\mathbb{R}_*} \phi(x) f(\mathrm{d}x) \right|$

We can integrate $\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_*} \phi(x) f(\mathrm{d}x, s)$ over the interval $[t_1, t_2]$. This gives us

$$\begin{aligned} &\int_{\mathbb{R}_*} \phi(x) f(\mathrm{d}x, t_2) - \int_{\mathbb{R}_*} \phi(x) f(\mathrm{d}x, t_1) \\ &= \frac{1}{2} \int_{t_1}^{t_2} \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} K(x, y)(\phi(x + y)\zeta_L(x + t) - \phi(x) - \phi(y)) f(\mathrm{d}x, s) f(\mathrm{d}x, s) \\ &+ \int_{t_1}^{t_2} \int_{\mathbb{R}_*} \phi(x) \eta(\mathrm{d}x) \mathrm{d}s \end{aligned}$$

From Lemma 5.7, it follows that

$$\begin{aligned} p_\phi(\alpha(t_2) - \alpha(t_1)) &= p_\phi(S(t_2)f_0 - S(t_1)f_0) \\ &= \left| \int_{\mathbb{R}_*} \phi(x)(f(\mathrm{d}x, t_2) - f(\mathrm{d}x, t_1) \right| \\ &\leq C(t_2 - t_1)\|\phi\|, \end{aligned}$$

where the constant C depends on the time-dependent measure f only via $\sup_{s\in[t_1,t_2]} \|f(s)\|$. Therefore, by Proposition 3.42, $\alpha\colon \mathbb{R}_+ \to \mathscr{X}_L$ is continuous when the target space is equipped with the weak*-topology. □

Next, we move on to consider how the semigroup $(S(t))_{t\in\mathbb{R}_+}$ evolves the total measures of the initial measure $f_0 \in \mathscr{X}_L$. To this end, for each $f_0 \in \mathscr{X}_L$, we denote $I(f_0) \coloneqq \int_{[1,2L]} f_0(\mathrm{d}x)$. In other words, our next task is to study the evolution of $I(S(t)f_0)$ when $t \in \mathbb{R}_+$ varies.

We now introduce the concept of *positive invariance* of a set, as it is defined in [30, p. 16–17].

Definition 5.11 (Positively invariant set)**.** Given a semigroup of operators $\{S(t)\}_{t\in\mathbb{R}_+}$ acting on a space Y, we call a set $X \subset Y$ positively invariant for the semigroup in case $S(t)X \subset X$ for every $t \geq 0$. ◁

Now, we want to find a set that is positively invariant with respect to the time evolution semigroup of operators.

Lemma 5.12. *Suppose that $\{S(t)\}_{t\in\mathbb{R}_+}$ is the semigroup of operators on $\mathscr{X}_L$ given by $S(t)f_0 = f(\cdot, t) \in \mathscr{X}_L$. Then, there exists a positively invariant set $\mathcal{U}_M \subset \mathscr{X}_L$ with respect to this semigroup. Moreover, $\mathcal{U}_M$ is non-empty, convex and compact in the weak*-topology.*

Proof. We recall that the time dependent solution associated with the semigroup $\{S(t)\}_{t\in\mathbb{R}_+}$ belongs to $f \in C^1([0,\infty), \mathscr{X}_L)$. Moreover, as seen from the derivation of (5.15), we can show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_*} f(\mathrm{d}x, t) \leq -\frac{1}{2} \int_{\mathbb{R}_*} f(\mathrm{d}y, t) \int_{\mathbb{R}_*} K(x, y) f(\mathrm{d}x, t) + \int_{\mathbb{R}_*} \eta(\mathrm{d}x).$$

Since $K(x,y) \geq a_1$, this yields the following estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_*} f(\mathrm{d}x,t) \leq -\frac{a_1}{2}\left(\int_{\mathbb{R}_*} f(\mathrm{d}x,t)\right)^2 + \int_{\mathbb{R}_*}\eta(\mathrm{d}x).$$

Since $f(\cdot,t)$ vanishes outside $[1,2L]$, this yields the following estimate.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{[1,2L]} f(\mathrm{d}x,t) \leq -\frac{a_1}{2}\left(\int_{[1,2L]} f(\mathrm{d}x,t)\right)^2 + \int_{\mathbb{R}_*}\eta(\mathrm{d}x). \tag{5.21}$$

If we use the notation

$$I(\mu) = \int_{[1,2L]}\mu(\mathrm{d}x), \quad \mu \in \mathcal{M}_b(\mathbb{R}_*),$$

it is evident that we can pick a large constant $M \geq 0$ so that the right hand size of (5.21) becomes less than or equal to 0 when $I(f(t))$ is close to M. Indeed, let us pick $M = \sqrt{\frac{4\|\eta\|}{a_1}}$. We can then show that

$$\mathcal{U}_M \coloneqq \{f_0 \in \mathscr{X}_L \colon I(f_0) \leq M\}$$

is an invariant region.

Note that our choice of M, we have $\frac{\mathrm{d}}{\mathrm{d}t}I(f(t)) < 0$ whenever $I(f(t)) > M/2$. Using this fact, we want to show that $f_0 \in \mathcal{U}_M$ is positively invariant under the action of the semigroup $\{S(t)\}_{t\in\mathbb{R}_+}$. Assume that $f_0 \in \mathcal{U}_M$. The positive invariance holds, in case

$$D_f \coloneqq \{t \in \mathbb{R}_+ \colon I(S(t)f_0) \leq M\} = \mathbb{R}_+,$$

since this is just another way of saying that $S(t)f_0 \in \mathcal{U}_M$ for every $t \in \mathbb{R}_+$.

Consider the trajectory $(S(t)f_0)_{t\in\mathbb{R}_+}$. Per our assumption, $f_0 \in \mathcal{U}_M$, so $0 \in D_f$. Suppose, then, that we have $t_0 \in D_f$, but $t_1 \notin D_f$ for some t_0, t_1 with $0 \leq t_0 < t_1$. Let $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ be the differentiable function given by $\phi(t) = I(S(t)f_0) \in \mathbb{R}$. Since $t_0 \in D_f$ and $t_1 \notin D_f$, we have $\phi(t_0) \leq M$ and $\phi(t_1) > M$. It follows that we can find a point $t^* \in [t_0, t_1)$ with $\phi(t^*) = M$ and $\phi'(t^*) \geq 0$. This contradicts the assumption that $I(f) = M$ implies $\frac{\mathrm{d}}{\mathrm{d}t}I(f(t)) < 0$. Consequently, $t \in D_f$ for each $t \geq 0$. Since f_0 was arbitrary measure in $\mathcal{U}_M$, this means that $S(t)\mathcal{U}_M \subset \mathcal{U}_M$ for any $t \in \mathbb{R}_+$.

To verify that the set $\mathcal{U}_M$ is compact, we first note that

$$\mathcal{U}_M \subset A \coloneqq \{f \in \mathcal{M}_b(\mathbb{R}_*) \colon \|f\| \leq M\},$$

and the set A is weak*-closed due to Banach-Alaoglu's theorem. Its topology is also metrizable. Therefore, in order to show that the set $\mathcal{U}_M$ is closed, we need to show that in contains all the limit points of sequences in $\mathcal{U}_M$. To this end, let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{U}_M$ be a sequence (in the weak*-topology, which is metrizable) that tends towards $f \in A$. Clearly $I(f) \leq M$. Moreover, since $\mathscr{X}_L$ is weak*-closed, the limit lies in $\mathscr{X}_L$, $f \in \mathscr{X}_L$. Therefore, $f \in \mathcal{U}_M$.

Establishing convexity of the set readily follows. For any $t \in [0,1]$, the positive linear combination

$$\mu_t = t\eta + (1-t)\nu$$

of two positive Radon measures $\eta, \nu \in \mathcal{M}_{+,b}(\mathbb{R}_*)$ is also a positive Radon measure. Moreover, if both η and ν vanish on the set $\mathbb{R}_* \setminus [1, 2L] = (0,1) \cup (2L, +\infty)$, then

$$\mu_t\left(\mathbb{R}_* \setminus [1,2L]\right) = t\eta\left(\mathbb{R}_* \setminus [1,2L]\right) + (1-t)\nu\left(\mathbb{R}_* \setminus [1,2L]\right) = 0.$$

Finally,

$$I(\mu_t) = I(t\eta + (1-t)\nu) \leq tM + (1-t)M \leq M.$$

Therefore, it follows that $\mu_t \in \mathcal{U}_M$. Since $\eta, \nu \in \mathcal{U}_M$ and $t \in [0,1]$ were arbitrary, the convexity of $\mathcal{U}_M$ follows. Since the empty measure lies inside $\mathcal{U}_M$, the set is non-empty.

In conclusion, $\mathcal{U}_M$ is a non-empty, closed and convex set that is invariant under the time evolution semigroup $\{S(t)\}_{t\in\mathbb{R}_+}$ □

Having shown that the set $\mathcal{U}_M$ is positively invariant with respect to the semigroup of operators $\{S_t\}$, we can use this to find a fixed point to this semigroup.

Lemma 5.13. *For $\phi \in C_0(\mathbb{R}_*)$, and for a weak solution f with $f(\cdot,t) \in \mathcal{U}_M$ for every $t_1, t_2 \in [0,T]$ with $t_2 \geq t_1$ we have*

$$\left|\int_{\mathbb{R}_*} \phi(x)\left(f(\mathrm{d}x, t_2) - f(\mathrm{d}x, t_1)\right)\right| \leq C_{M,\phi}(t_2 - t_1), \tag{5.22}$$

Proof. This is essentially established in the proof of Lemma 5.10. □

Remark. If we assume that $f \in C([0,T], \mathscr{X}_L)$ instead of $f(\cdot,t) \in \mathcal{U}_M$, then the constant depends on T via the norm $\|f\|_T < +\infty$.

Proposition 5.14. *Let $\{S(t)\}_{t\in\mathbb{R}_+}$ be the semigroup of operators defined for each $t \geq 0$ by*

$$\begin{aligned} S(t)\colon \mathcal{U}_M &\to \mathcal{U}_M \\ f &\mapsto f(\cdot, t), \end{aligned}$$

where $\mathcal{U}_M$ is the set defined in Lemma 5.12. In other words, this is the restriction of our previous semigroup to $\mathcal{U}_M \subset \mathscr{X}_L$.

There exists a measure $\hat{f} \in \mathcal{U}_M$ such that $S(t)\hat{f} = \hat{f}$ for every $t \in \mathbb{R}_+$.

Proof. We know that $\mathcal{U}_M \subset \mathcal{M}_{+,b}(\mathbb{R}_*)$ is non-empty, convex and weak*-compact, and that for every $r > 0$, the map $S(r)\colon \mathcal{U}_M \to \mathcal{U}_M$ is weak*-continuous. This allows us to use Schauder's fixed-point theorem, which gives a fixed point for every $r > 0$. We denote this fixed point of the map $S(r)$ by $\hat{f}_r$. Thus, $S(r)\hat{f}_r = \hat{f}_r \in \mathcal{U}_M$.

Since $\mathcal{U}_M$ is compact, $\mathcal{M}_b(\mathbb{R}_*) \supset \mathcal{U}_M$, and $\mathcal{M}_b(\mathbb{R}_*) \cong C_0(\mathbb{R}_*)^*$ where $C_0(\mathbb{R}_*)$ is separable, it follows from Theorem 3.44 that the weak*-topology on $\mathcal{U}_M$ is metrizable. This implies that $\mathcal{U}_M$ is sequentially compact. Thus, we can find sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ such that $\varepsilon_i \to 0$ and the sequence of measures $(\hat{f}_{\varepsilon_i})$ converges to some $\hat{f} \in \mathcal{U}_M$ in the weak*-topology.

Note that since $\hat{f}_{\varepsilon_i} \in \mathcal{U}_M$ for every i, it follows from (5.22) that we have the uniform estimate

$$\left|\left\langle \phi, S(r)\hat{f}_{\varepsilon_i} - \hat{f}_{\varepsilon_i} \right\rangle\right| \leq rC_{M,\phi}.$$

Let $t \in \mathbb{R}_+$. We want to show that this $\hat{f}$, which does not depend on t, is a fixed point of the operator $S(t)$.

For every $i \in \mathbb{N}$, we can find $n = n(\varepsilon_i, t) \in \mathbb{N}$ satisfying the estimates $(n-1)\varepsilon_i \leq t < n\varepsilon_i$. Therefore, we can find a function $l_t\colon \mathbb{R} \to \mathbb{R}_+$ satisfying $l_t(\varepsilon_i) \leq \varepsilon_i$ and $t = (n-1)\varepsilon_i + l_t(\varepsilon_i)$, where n satisfies the above estimates. In particular, we can assume that $l_t(\varepsilon_i) < 1$ for every $i \in \mathbb{N}$. By the semigroup property of the family $\{S(t)\}_{t\in\mathbb{R}_+}$, we have $S(t)\hat{f}_{\varepsilon_i} = S(l_t(\varepsilon_i) + (n-1)\varepsilon_i)\hat{f}_{\varepsilon_i} = S(l_t(\varepsilon_i))\hat{f}_{\varepsilon_i}$.

Pick a test function $\phi \in C_0(\mathbb{R}_*)$. The triangle inequality and the uniform estimate (5.22) tell us that

$$\begin{aligned} \left|\left\langle \phi, S(l_t(\varepsilon_i))\hat{f}_{\varepsilon_i} - \hat{f} \right\rangle\right| &\leq \left|\left\langle \phi, S(l_t(\varepsilon_i))\hat{f}_{\varepsilon_i} - \hat{f}_{\varepsilon_i} \right\rangle\right| + \left|\left\langle \phi, \hat{f}_{\varepsilon_i} - \hat{f} \right\rangle\right| \\ &\leq l_t(\varepsilon_i)C_{M,\phi} + \left|\left\langle \phi, \hat{f}_{\varepsilon_i} - \hat{f} \right\rangle\right| \xrightarrow{i\to\infty} 0. \end{aligned}$$

Consequently, $S(t)\hat{f}_{\varepsilon_i} = S(l_t(\varepsilon_i))\hat{f}_{\varepsilon_i} \to \hat{f}$ in the weak*-topology. On the other hand, by the continuity of $S(t)$ for a fixed t, we have $S(t)\hat{f}_{\varepsilon_i} \xrightarrow{i\to\infty} S(t)\hat{f}$. It follows from the uniqueness of weak*-limits that $S(t)\hat{f} = \hat{f}$. Since this holds for every $t \in \mathbb{R}_+$, and the constructed measure $\hat{f}$ does not depend on t, it follows that $\hat{f}$ is stationary with respect to the semigroup $\{S(t)\}_{t\in\mathbb{R}_+}$. □

We have shown that given any compactly supported injection measure η and a good coagulation kernel K, where the family of good coagulation kernels depends on η, there exists an initial condition f_0 such that $S(t)f_0 = f_0$ for every $t \geq 0$. Here $\{S(t)\}_{t\in\mathbb{R}_+}$ evolves f_0 according to the unique solution to the time-dependent regularized coagulation equation with f_0 as the initial value. It follows that the constant measure f_0 satisfies Definition 4.6.

In order to see that f_0 is a nontrivial stationary solution, it suffices to pick some test function $\phi \in C_c(\mathbb{R}_*)$ with $\phi \succ \mathrm{supp}(\eta)$[9], and $0 \leq \phi \leq 1$. Assuming that $f_0 = 0$ then yields the inequality $\eta(\mathrm{supp}(\eta)) \leq \int_{\mathbb{R}_*} \phi(x)\eta(\mathrm{d}x) = 0$. This is in contradiction with the assumption that η is a non-trivial positive measure. Therefore, $f_0 \neq 0$. This concludes the proof of Theorem 4.7.

[9]See Appendix A for the notation.

6 Discussion

In this master's thesis, we have demonstrated how regularized coagulation equations with injection have time-independent solutions, provided that the coagulation kernel belongs to a suitable family of functions. The construction of kernel and the regularization might seem a bit artificial, and indeed they are. The main point of the theorem is to act as a stepping stone towards establishing the existence of stationary solutions to (no longer regularized) coagulation equation with injection and kernels that behave asymptotically in a certain way.

This move from regularized coagulation equations with compactly supported kernels to coagulation equations with asymptotically nicely behaving kernels will be discussed in the next subsection. The full proof, which can be found in [6], is beyond the immediate scope of this thesis, so we will only outline the main ideas of the proof. After giving a high-level view of the theorem and its proof, we will mention a complementary result that allows us to classify those coagulation kernels for which no stationary solutions exist. Finally, we make some notes on possible further work to be done related to the theorem presented in this thesis.

6.1 From compactly supported kernels to asymptotically well-behaving kernels

We use the notation $g \sim f$ to mean that $cf \leq g \leq Cf$ for some constants $c, C > 0$. As already mentioned, the existence theorem can be extended to cover all coagulation kernels K satisfying the asymptotic law

$$K(x, y) \sim x^{\gamma+\lambda} y^{-\lambda} + y^{\gamma+\lambda} x^{-\lambda}$$

with $|\gamma + 2\lambda| < 1$. The proof is computationally more involved than the proof considered in this thesis, which is part of the reason for not proving it here. This claim is proved in detail in by Ferreira et al. in [6], but we will give a high-level sketch of it here, since this sheds light on why we were interested in the regularized problem with compactly supported kernels in the first place. The statement of the theorem goes as follows.

Theorem 6.1 (Existence of solutions [6])**.** *Suppose that* $K \colon \mathbb{R}_* \times \mathbb{R}_* \to \mathbb{R}_+$ *is a continuous, positive and symmetric map. Assume further that*

$$K(x, y) \sim x^{\gamma+\lambda} y^{-\lambda} + y^{\gamma+\lambda} x^{-\lambda}, \tag{6.1}$$

where the constants $\lambda, \gamma \in \mathbb{R}$ satisfy the additional requirement that $|\gamma + 2\lambda| < 1$. Then, for a given $\eta \in \mathcal{M}_{+,b}(\mathbb{R}_*) \setminus \{0\}$ supported on the compact interval $[1, C_\eta]$, there exists a stationary solution $f \in \mathcal{M}_+(\mathbb{R}_*) \setminus \{0\}$ satisfying the coagulation equation with injection.

Remark. Remark that stationary solution need not be bounded, i.e. it can belong to $\mathcal{M}_+(\mathbb{R}_*) \setminus \mathcal{M}_{+,b}(\mathbb{R}_*)$.

To understand how the theorem might be proved, we first note that any kernel behaving asymptotically like (6.1) can be decomposed as a product

$$K(x, y) = (x + y)^\gamma \Phi\left(\frac{x}{x+y}, x\right).$$

Here, the factor $\Phi\colon \mathbb{R}_* \times \mathbb{R}_* \to \mathbb{R}_+$ is a continuous function satisfying the asymptotic behaviour

$$\Phi(s, x) \sim \frac{1}{s^p(1-s)^p}, \tag{6.2}$$

where the exponent is given by $p = \max\{\lambda, -(\gamma + \lambda)\}$. We shall denote by c and C the bounding constants of $\Phi(s, x)$, and these constants are independent of x.

We can truncate the kernel K in two levels to arrive at a function K_{ε,R_*}, which is compactly supported and satisfies the assumptions laid out in Theorem 4.7. Therefore, we find a measure-value function f_{ε,R_*}, which is a stationary solution to the injection coagulation equation with the kernel K_{ε,R_*}.

We can then use test functions to find a constant C_ε satisfying

$$\int_{\mathbb{R}_*} f_{\varepsilon,R_*}(\mathrm{d}x) \leq C_\varepsilon, \tag{6.3}$$

and which does not depend on the truncation value R_*. We can then integrate f_{ε,R_*} over $\mathbb{R}_*$ to get a the same upper bound for this integral. Moreover, we get the tail bounds

$$\int_{[y,+\infty)} f_{\varepsilon,R_*}(\mathrm{d}x) \leq C_\varepsilon y^{-1/2}. \tag{6.4}$$

If we let $(R_*^n)_{n\in\mathbb{N}}$ be a sequence tending to infinity, then we can use the upper bound (6.3) to get a limit $f_\varepsilon \in \mathcal{M}_{+,b}(\mathbb{R}_+)$ with $f_\varepsilon((0,1)) = 0$ and

$$f_{\varepsilon,R_*^n} \to f_\varepsilon, \text{ in the weak*-topology.}$$

It is easily verified that this limit satisfies bounds (6.3) and (6.4).

Next, we note that $K_{\varepsilon,R_*} \to K_\varepsilon$, as $R_* \to \infty$, where K_ε is a suitably defined kernel, and the limit measure f_ε is easily verified to satisfy the identity

$$0 = \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_\varepsilon(x,y)\left(\phi(x+y)-\phi(x)-\phi(y)\right) f_\varepsilon(\mathrm{d}x) f_\varepsilon(\mathrm{d}y) + \int_{\mathbb{R}_*}\phi(x)\eta(\mathrm{d}x) \tag{6.5}$$

Moreover, the bound (6.3) implies that

$$\int_{[0,\infty)} f_\varepsilon(\mathrm{d}x) \le C_\varepsilon. \tag{6.6}$$

This and other bounds allow us to get estimates on the behavior of f_ε, which establishes weak*-compactness of the family $\{f_\varepsilon\}_{\varepsilon>0}$. Moreover, we can show that the modified measures $x^{\gamma+p}f$ lie inside a weak*-compact set. It follows that there exists a measure $F \in \mathcal{M}_{+,b}(\mathbb{R}_+)$ and a subsequence ε_n such that

$$x^{\gamma+p} f_{\varepsilon_n} \to F, \text{ in the weak*-topology as } n \to \infty.$$

The measure defined by letting $f = x^{-(\gamma+p)}F$ lies inside $\mathcal{M}_+(\mathbb{R}_*)$. If $\gamma > 0$, it follows that $f \in \mathcal{M}_{+,b}(\mathbb{R}_*)$ and $f_{\varepsilon_n} \to f$ in the weak*-topology.

Next, with considerable care (and here we use the crucial assumption that $|\gamma + 2\lambda| < 1$), we can show that

$$\int_{\mathbb{R}_*^2} K(x,y)\left(\phi(x+y)-\phi(x)-\phi(y)\right) f(\mathrm{d}x) f(\mathrm{d}y) + \int_{\mathbb{R}_*}\phi(x)\eta(\mathrm{d}x) = 0 \tag{6.7}$$

for every $\phi \in C_c(\mathbb{R}_*)$. Moreover, we get μ-moment bounds for f, where $\mu < \frac{\gamma+1}{2}$. The assumption $|\gamma + 2\lambda| < 1$ then implies that the moment bounds hold for $-\lambda$ and $\lambda + \gamma$. This concludes the proof.

Examples of kernels that satisfy the assumptions of the theorem are the constant kernel $K(x,y) \equiv 1$ and the Brownian kernel, where the latter one was discussed in Section 2.

6.2 Non-existence of non-trivial stationary solutions for certain kernels

In the previous subsection, we noted that for coagulation kernels behaving asymptotically like (6.1) with $|\gamma + 2\lambda| < 1$, every injection measure yields a stationary solution to the

coagulation equation. It turns out that the assuming $|\gamma + 2\lambda| < 1$ is crucially part of this result. Indeed, Ferreira et al. show in [6] that in case the coagulation kernel has the above asymptotic behaviour (6.1), but this time with λ and γ satisfying the condition $|\gamma + 2\lambda| \geq 1$, then we have quite a different result. In this case, no non-trivial injection measure has a stationary solution to the coagulation equation. The non-existence proof by contradiction uses very different techniques than the existence theorem, and we will not give an outline of it. Instead, we refer the interested reader to Section 4 in [6].

The upshot of this is the following characterization: suppose that the coagulation kernel behaves like (6.1) with parameters $\lambda, \gamma \in \mathbb{R}$. Then precisely one of the following two conditions holds

(C1) The parameters γ and λ satisfy $|\gamma + 2\lambda| < 1$, or

(C2) the parameters satisfy $|\gamma + 2\lambda| \geq 1$.

In the first case, for any nonzero injection measure there exists a nontrivial stationary solution. In the second case, no nonzero injection measure admits a stationary solution. Therefore, whether a stationary solution exists is completely characterized by the above two possibilities. In other words, checking whether stationary solutions exists for a kernel behaving asymptotically like (6.1) is reduced to checking which one of the conditions (C1) and (C2) holds.

6.3 Discrete coagulation equation with injection

As mentioned above, discrete results analogous to the continuous ones discussed in the above sections also hold. We define discrete stationary solutions to coagulation equations with injection as follows.

Definition 6.2. Suppose that $K\colon \mathbb{N}^2 \to \mathbb{R}_+$ is a symmetric function satisfying the asymptotic behaviour

$$K_{\alpha,\beta} \sim \alpha^{\gamma+\lambda}\beta^{-\gamma} + \beta^{\gamma+\lambda}\alpha^{-\gamma},$$

and $s_\alpha\colon \mathbb{N} \to \mathbb{R}_+$ is a sequence supported in $\mathbb{N} \cap [1, L]$ for some L. Let $n\colon \mathbb{N} \to \mathbb{R}_+$ be a sequence with

$$\sum_{\alpha=1}^{\infty} \alpha^{\gamma+\lambda} n_\alpha + \sum_{\alpha=1}^{\infty} \alpha^{-\lambda} n_\alpha < +\infty.$$

This is called a stationary solution of the discrete coagulation equation with injection, in case for any compactly supported test sequence $\phi\colon \mathbb{N} \to \mathbb{R}$, we have

$$\frac{1}{2}\sum_{\beta=1}^{\infty}\sum_{\alpha=1}^{\infty} K_{\alpha,\beta} n_\alpha n_\beta \left(\phi_{\alpha+\beta} - \phi_\alpha - \phi_\beta\right) + \sum_{\beta=1}^{\infty} s_\beta \phi_\beta = 0 \tag{6.8}$$

◁

As proven in [6], the following two theorems hold.

Theorem 6.3. *Let $K\colon \mathbb{N}^2 \to \mathbb{R}_+$ be a coagulation kernel with the asymptotic exponents γ and λ satisfying $|\gamma + 2\lambda| < 1$. Let $s\colon \mathbb{N} \to \mathbb{R}_+$ be a nonzero compactly supported injection sequence. Then, there exists a nontrivial sequence $n\colon \mathbb{N} \to \mathbb{R}$, which is a stationary solution in the sense of Definition 6.2.*

Theorem 6.4. *Let $K\colon \mathbb{N}^2 \to \mathbb{R}_+$ be a coagulation kernel with the asymptotic exponents γ and λ satisfying $|\gamma + 2\lambda| \geq 1$. Let $s\colon \mathbb{N} \to \mathbb{R}_+$ be a nonzero compactly supported injection sequence. Then, there exists no sequence $n\colon \mathbb{N} \to \mathbb{R}$ that is a stationary solution in the sense of Definition 6.2.*

Full proofs of these theorems can be found in [6]. The proof of Theorem 6.3 is essentially the same as the proof of the continuous existence theorem, and can be readily understood after going through the proof of the continuous case.

On the other hand, the proof of Theorem 6.4 is quite short once the continuous version is established. The main idea behind it is that assuming that there exists a solution satisfying the assumption yields a contradiction with the continuous nonexistence theorem. To be more precise, let us assume that for a given kernel $K\colon \mathbb{N}^2 \to \mathbb{R}_+$ and an injection sequence $s\colon \mathbb{N} \to \mathbb{R}_+$ there exists a stationary solution $n\colon \mathbb{N} \to \mathbb{R}$ in the sense of Definition 6.2. We can then define a corresponding "continuous" interpolation kernel function $\tilde{K}\colon \mathbb{R}_* \times \mathbb{R}_* \to \mathbb{R}_+$, which agrees with K on the points with natural number coordinates. If we also define a new injection measure $\eta = \sum_{\alpha=1}^{\infty} \delta_\alpha s_\alpha$ and a candidate for a stationary solution $f = \sum_{\alpha=1}^{\infty} \delta_\alpha n_\alpha$, then both measures are in $\mathcal{M}_+(\mathbb{R}_*)$ and f can be verified to satisfy the continuous coagulation equation with the kernel $\tilde{K}$ and the injection measure η. But this yields a contradiction with the nonexistence theorem in discussed in the previous subsection.[10]

[10] See Theorem 2.4 in [6].

6.4 Further work

We have seen that Theorem 4.7 says nothing about the uniqueness of the stationary solution. The reason for this is that the proof evokes Schauder's fixed point theorem, which gives the existence of fixed points. However, these might not be unique. As is usual in the study of integral and differential equations, being able to say that a solution is unique or not necessarily unique is desirable, as it improves our understanding of the problem and helps us connect the mathematical model to the behaviour of the physical process that we are studying. The question then arises as to whether we can improve the theorem to show that the solutions are unique, or whether we can give a construction that there are nonunique stationary solutions. To show the uniqueness of solutions, one might want to circumvent the use of Schauder's fixed point theorem or else use the definition of stationary solutions to derive a contradiction assuming that two stationary solutions exist. This might be tricky, though, given the nonlinear structure of the regularized coagulation equation.

Another topic of interest is the regularity of stationary solutions. Recall that we assumed the coagulation rate function to be continuous on $\mathbb{R}^2_*$. It can be shown that for an injection measure with continuous density, the resulting stationary solution also has a continuous density. Therefore, stationary solutions cannot be singular with respect to a continuous initial measure. This proven in [6]. Provided that the injection rate function and the coagulation kernel are k-smooth and we have additional upper bounds on the derivatives of K up to the kth order, this proof of this can be iterated to show that smooth injection measures yield smooth stationary solutions.

On the other hand, it isn't true that the solution measure has to be absolutely continuous with respect to the injection measure. As a simple example, if the injection measure is δ_1, then we can find a kernel K which gives a stationary solution with a strictly larger support than δ_1, whereby it is not absolutely continuous with respect to the injection measure.

The combination of the questions of uniqueness and regularity yields another research question. Namely, provided that the stationary solutions are not unique, can we at least establish uniqueness in certain regularity classes. One interesting question around this theme is whether we can prove that continuous stationary solution are unique – provided that they exist.

Appendices

Appendix A Notation

- $\mathbb{N} = \{1, 2, 3, 4, \dots\}$, the set of natural numbers (without zero).
- $\mathbb{R}_+ = [0, +\infty)$ the set of non-negative real numbers.
- $\mathbb{R}_* = \mathbb{R}_+ \setminus \{0\} = (0, +\infty)$, the set of strictly positive real numbers.
- X', the algebraic dual of X, i.e. the set of all linear functionals $\lambda \colon X \to \mathbb{K}$ where $\mathbb{K}$ is the underlying field of X.
- $X^* \coloneqq \{\Lambda \colon X \to \mathbb{K} \mid \Lambda \text{ is continuous } C(X, \mathbb{K})\}$. This is known as the topological dual space of X.
- $X^{**} \coloneqq (X^*)^*$, the double dual of X, that is, the topological dual of X^*
- $\sigma(X, X^*)$, the topology on X induced by the family of maps X^*. This is the weak*-topology on X.
- $\mathcal{M}_+(X)$, the collection of positive Radon measures defined on X.
- $\mathcal{M}_b(X)$, the collection of signed, bounded Radon measures defined on X.
- $\mathcal{M}_{+,b}(X) = \mathcal{M}_b(X) \cap \mathcal{M}_+(X)$, the collection of positive, bounded Radon measures on X.
- $\mathcal{B}(X, Y)$: the space of bounded linear operators between spaces X and Y.
- $C(X, Y)$, the collection of continuous functions $f \colon X \to Y$.
- $C^1([0, \infty), S)$, where $S \subset Y$ and Y is a normed space, is the collection of continuous functions $f \colon [0, \infty) \to S$ that have a continuous Fréchet derivative $\dot{f} \colon [0, \infty) \to Y$. Similarly for $f \in C^1([0, T], S)$.
- $C^k(X, Y)$, the collection of continuous, k times continuously differentiable functions $X \to Y$.

- $C_c(X)$, the collection of all continuous, compactly supported functions $X \to \mathbb{R}$.
- $C_0(X)$, the completion of $C_c(X)$ in the space $C_b(X)$ of bounded continuous functions equipped with the topology given by the supremum norm.
- $f \prec A$, where $A \subset X$, means that f is a positive function on X which is supported on A.
- $f \succ A$, where $A \subset X$, means that f is a positive function that takes value 1 on A.
- $f \sim g$, where f and g are real valued functions on X, means that f and g behave similarly. That is, there exists $C_1, C_2 > 0$ such that $C_1 f(x) \leq g(x) \leq C_2 f(x)$ for every $x \in X$.

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